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Mathematical Reviews

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R. P. Boas, Jr., *Executive Editor*

Vol. 8, No. 10 November, 1947 pp. 557–616

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MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
THE INSTITUTE OF MATHEMATICAL STATISTICS
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Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence 12, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence 12, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorized November 9, 1940.

Mathematical Reviews

Vol. 8, No. 10 NOVEMBER, 1947 Pages 557-616

FOUNDATIONS

*Reichenbach, Hans. **Elements of Symbolic Logic.** The Macmillan Company, New York, N. Y., 1947. xv+444 pp. $5.00.

This textbook, not primarily intended for mathematicians, emphasizes more the interpretation and application of formal logic than the construction of the formal system itself. Logical expressions are considered throughout as having a meaning; in particular, existence is interpreted as meaning physical existence. Everywhere the author's extreme physicalist position in philosophy is explicit. The introductory chapter I treats of the nature and usage of language. Chapter II contains an exposition of the calculus of propositions, based on truth-tables. Noteworthy here is the distinction between adjunctive and connective interpretation of the logical signs, corresponding to the well-known distinction between intensional and extensional concepts; the former distinction is extended to all logical signs and finds an ample treatment in the last chapter [VIII] of the book. Chapters III and IV treat the simple calculus of functions. The author starts by giving a list of formulas which can be proved tautological by considering the three possible values "always true," "always false," "sometimes true, sometimes false" for the functions which occur. Then he describes the process of derivation and proves that it leads to no contradiction. Chapter V is on the theory of classes. In chapter VI the theory of functions of higher types is treated. The author's philosophical position leads to peculiar difficulties concerning the meaning of existence in this field. The usual antinomies are treated and solved by the simple theory of types, completed by the theory of levels of language. In treating the technique of this calculus, the author gives up the formal point of view by defining a tautology as "a true formula which contains no empirical constants, or a formula resulting from such a formula by specialization." For this reason, some of his considerations, e.g., the indications on Gödel's result about the incompleteness of the calculus, are rather unsatisfactory from the mathematical point of view. Chapter VII contains a lengthy discussion on the reform of grammar by means of formal logic. *A. Heyting* (Amsterdam).

Moisil, Gr. C. Sur la structure algébrique de la logique de M. Bochvar. Disquisit. Math. Phys. 1, 307–314 (1941).

A partir de la lattice à trois éléments L_3, définie par $a<f<v$ (f=faux, v=vrai), l'auteur définit les foncteurs T (négation) et D (démontrabilité) de Bochvar [Rec. Mãth. [Mat. Sbornik] N.S. (4)46, 287–308 (1938)]. Par un système d'axiomes involvant la conjonction, la disjonction, T et D il définit les algèbres bochvariennes trivalentes ou algèbres B_3. Toute algèbre B_3 peut être représentée par une isomorphie sur L_3^E, ou E est un ensemble convenablement choisi. Il obtient des résultats analogues sur la lattice L_4 ($a<f<p<v$) et les algèbres dites B_4. La lattice $f<p<v$ a des rapports analogues avec la logique tripartite de Łukasiewicz, comme l'auteur a montré autrefois [Ann. Sci. Univ. Jassy. Sect. I. 26, 431–436 (1940); 27, 86–98 (1941); ces Rev. 8, 307]. *A. Heyting* (Amsterdam).

Moisil, Gr. C. Logique modale. Disquisit. Math. Phys. 2, 3–98 (1942).

Aux axiomes I–III pour la logique positive selon Hilbert et Bernays [Grundlagen der Mathematik, t. 1, Springer, Berlin, 1934, p. 66] on adjoint un foncteur Sxy, satisfaisant à l'axiome $CyASyxx$ et à la règle: "si $CyAzx$, alors $CSyxz$"; c'est-à-dire S est une résiduation par rapport à la disjonction A dans la lattice considérée. Cxx est le dernier élément (le "vrai"), Sxx le premier élément (le "faux"). Les modalités sont introduites comme suit: impossibilité: $\eta x = CxSxx$; contingence: $\gamma x = SCxxx$; possibilité: $\eta\eta x$; nécessité: $\gamma\gamma x$. L'auteur étudie dans les détails cette "logique modale générale" et examine les conséquences de divers axiomes supplémentaires. Quelques résultats: (1) Les formules de la logique intuitioniste sont les formules de la logique modale générale ne contenant que les foncteurs C, A, K, η. (2) Par l'adjonction des axiomes (6.1) $CCKxyzACxzCyz$, (8.1) $CKSxxSxySxAxy$ et de l'un des axiomes 9I–9XI, par exemple (9XI) $AxCxy$, on obtient la logique classique. (3) En introduisant $Nx = A\eta xKx\gamma x$ et $Lx = ACxyCNyNx$ on obtient la logique trivalente de Łukasiewicz (avec L comme implication). *A. Heyting* (Amsterdam).

Quine, W. V. The problem of interpreting modal logic. J. Symbolic Logic 12, 43–48 (1947).

An introductory section deals with analytical statements from the standpoint of nonmodal logic. Analyticity is defined in terms of synonymy and logical truth as follows: a statement is analytic if by putting synonyms for synonyms it can be turned into a logical truth. The mode of logical necessity, symbolized by '$\Box$,' may then be explained by means of the rule (i) the result of prefixing '$\Box$' to any statement is true if and only if the statement is analytic. Quantification in modal logic is explained by means of the rule (ii) an existential quantification holds if there is a constant whose substitution for the variable of quantification would render the matrix true.

These rules, however, lead to paradoxical consequences. We have, indeed:

(1) $(\exists x)(x \subset \text{Evening Star}. \Box(x \subset \text{Morning Star}))$,

(2) $(\exists x)(x \subset \text{Evening Star}. \sim\Box(x \subset \text{Morning Star}))$.

The x whose existence is affirmed in (1) and the x whose existence is affirmed in (2) are two objects; so there must be at least two objects such that $x \subset$ Evening Star. Here '$\subset$' denotes the relation which exists between Venus, the Evening Star and the Morning Star and which is interpreted by materialistic astronomy as identity. Quine mentions the following alternatives to his interpretation. (a) We may regard quantified modal logic as only a part of a total logic, comparable to the one constructed by A. Church [same J. 11, 31 (1946)]. (b) We may reject the criterion (ii).

In the reviewer's opinion, however, we could just as well question the validity of the criterion (i). *E. W. Beth.*

***Skolem, Th. The development of recursive arithmetic.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 1–16. Jul. Gjellerups Forlag, Copenhagen, 1947.

This is an account of the theory of the different kinds of recursive functions, up to and including the general recursive functions of Herbrand-Gödel-Kleene. The emphasis is on the existence and decidability theorems and the relation of the different kinds of recursiveness both to one another and to other criteria of constructibility. The paper is similar in general content to the author's expository paper [Norsk Mat. Tidsskr. **28**, 1–12 (1946); these Rev. **8**, 125] but differs in the amount of space devoted to the various topics; in fact each of the two papers gives details not found in the other. The present paper also refers to more recent work. [One reference is erroneous; the fifth paper cited on p. 115 is in Amer. J. Math. **63**, 263–282 (1940), not Bull. Amer. Math. Soc.] Like the previous paper, this one is interesting for the author's slant on various topics, and direct, simple proofs of certain of the theorems. *H. B. Curry.*

Post, Emil L. Recursive unsolvability of a problem of Thue. J. Symbolic Logic **12**, 1–11 (1947).

The problem of Thue proved to be recursively unsolvable is that of deciding whether two words (finite "strings of letters") are equivalent under repeated applications of a given finite set of substitutions $A_i \rightleftarrows B_i$ (i.e., PA_iQ can replace PB_iQ and vice versa, where A_i and B_i are words from a given set of pairs, and P and Q are arbitrary words, possibly empty) [Thue, Skr. Videnskapsselsk. Kristiania. I. **1914**, no. 10]. The symbols $q_iS_jLq_l$, $q_iS_jRq_l$, $q_iS_jS_kq_l$ that describe the configurations of a Turing machine [Turing, Proc. London Math. Soc. (2) **42**, 230–265 (1936)] are made the basis for the construction of a "Thue system," the allowed substitutions corresponding to pairs of symbols that describe a configuration X and the configuration Y that succeeds X when the machine operates. (The relation between X and Y is not symmetrical in the machine, but in the corresponding "Thue system" this asymmetry is removed.) The known impossibility of finding a recursive process for deciding whether a symbol S_j will ever be printed by a given machine [Turing, op. cit.] leads to the example sought.

A four-page appendix contains a "critique" of Turing's paper [cited above], giving the author's reasons for relying on results of his own [Post, Bull. Amer. Math. Soc. **50**, 284–316 (1944); **52**, 264–268 (1946); these Rev. **6**, 29; **7**, 405], rather than making direct use of Turing's theorems. The permanence of the symbols '0' and '1' when once printed, and the distinction between E- and F-squares, introduced by Turing as "conventions," really form part of the definition of a machine, as used in his proofs. This changes the content of the theorems, so that they are not ready for use in the present context without modification. *M. H. A. Newman* (Manchester).

Markoff, A. On the impossibility of certain algorithms in the theory of associative systems. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 583–586 (1947).

Two theorems on recursive unsolvability in associative systems with a finite set of generators and relations are stated, and proved in outline. Theorem 1 asserts the unsolvability of the word (identity) problem and is identical with a theorem of E. L. Post [see the preceding review]. The method of proof is essentially similar, making use of an earlier result of Post [Amer. J. Math. **65**, 197–215 (1943); these Rev. **4**, 209] and Rosser's combinatorial logic [Ann. of Math. (2) **36**, 127–150 (1935)]. Theorem 2 states that an associative system can be constructed in which the word problem is solvable but the division problem is unsolvable, i.e., the question whether $XQ \leftrightarrow R$ has a solution in the system for any given Q and R. Let A be a finite alphabet and G_i, G_i' $(i=1, 2, \cdots, m)$ words in it such that the word problem is unsolvable under the one-way substitution laws "$G_iP \rightarrow PG_i'$ for any word P" [Post, op. cit. (1943)]. Then a system for theorem 2 is obtained by adding the new letters d, f, e_i to the alphabet and taking as relations $e_if \leftrightarrow fe_i$, $e_iG_i\alpha \leftrightarrow \alpha e_iG_i$, $e_iG_id \leftrightarrow G_i'd$ $(\alpha \epsilon A,\ i=1, \cdots, m)$. *M. H. A. Newman* (Manchester).

Kalmár, László. Ein einfaches Beispiel für ein unentscheidbares arithmetisches Problem. Mat. Fiz. Lapok **50**, 1–23 (1943). (Hungarian. German summary)

Kalmár gives what he considers a simplified proof of the famous incompleteness theorems of Gödel [Monatsh. Math. Phys. **38**, 173–198 (1931)]. The presentation differs from that of Gödel in two respects. First, Kalmár's arithmetical functions are defined, not recursively, but explicitly in terms of the functions $a+b$, ab, $|a-b|$ (absolute value of $a-b$) $[a/b]$ (integral part of a/b) together with sums and products $\sum_{z=a}^{b} f(x)$ and $\prod_{z=a}^{b} f(x)$, where $f(x)$ is a function already defined. [The class of functions so generated he calls elementary functions; he does not discuss their relations to the primitive recursive functions except to show, in a footnote, that they are included in the latter.] Secondly, Kalmár does not formulate the conditions in his system with the same explicitness and carries through the details only under certain simplifying assumptions; this method has the advantage of making the generality and essential nature of the theorem stand out more clearly, but of course the reader would have to supply many details in order to reproduce Gödel's proof in its original form. *H. B. Curry.*

Bing, Kurt. On Germansky's system of axioms for the natural numbers. II. Riveon Lematematika **1**, 57–60 (1946). (Hebrew)

The author concludes the proof of the equivalence of the Peano and Germansky axioms [Riveon Lematematika **1**, 21–28 (1946); these Rev. **8**, 126]. In addition the independence of the Germansky axioms is discussed. *S. Eilenberg* (New York, N. Y.).

Borel, Émile. Sur l'illusion des définitions numériques. C. R. Acad. Sci. Paris **224**, 765–767 (1947).

Pour faire suite à son ouvrage "Les Paradoxes de l'Infini" [Gallimard, Paris, 1946], l'auteur explique que, lorsque le fini devient trop grand, il soulève les mêmes difficultés que l'infini. Bien qu'on puisse, par exemple, définir un nombre entier très grand de manière univoque, une telle définition est illusoire, puisqu'elle ne permette pas d'en connaître une propriété mathématique autre que sa définition.

L'auteur se voit ainsi amené à faire une distinction entre définitions logiques et définitions d'approximation arithmétique. Une définition logique, préférable au point de vue purement théorique, n'a pas l'intérêt pratique qu'on serait incliné à lui prêter; encore même qu'une telle définition entraîne une méthode d'approximation numérique successive, on ne sera pas en état de continuer cette approximation de façon indéfinie. D'autre part, une définition d'approximation arithmétique n'a aucun intérêt théorique, si cette

définition ne puisse pas se présenter sous forme analytique; dans ce cas-là, cependant, elle équivaudrait à une définition logique. Il y a, par conséquent, une disparité complète entre les exigences de la théorie et celles de la pratique. Il ne faut pas s'arrêter à l'objection qu'il n'est pas possible de fixer une limite exacte pour la continuation d'une approximation numérique; cette objection n'est autre que le sophisme du tas de blé. [Des idées tout à fait semblables ont été soutenues par G. F. C. Griss, Idealistische Filosofie, Arnhem, 1946, pp. 23–24.] *E. W. Beth* (Amsterdam).

Pompeiu, D. Réflexions sur l'arithmétique de l'infini. Bull. Sci. École Polytech. Timişoara 11, 14–21 (1943).

Schrödinger, Erwin. The foundation of the theory of probability. I. Proc. Roy. Irish Acad. Sect. A. 51, 51–66 (1947).

The paper attempts to furnish axiomatic foundations for the probability of events, that is, of precise and unambiguous statements about the real world, statements the truth or falsity of which can in principle be determined by observation. Frequencies are not used; instead, the position is taken that probability is a number with which we try (in some cases) to set up a quantitative measure of the strength of our conjecture or anticipation (founded on a body of knowledge) that the event in question is true. The position thus belongs to the epistemological category of the "degree of rational belief" of the older authors, of "degree of confirmation" [Nagel, Carnap, etc.], and of "intuitive probability" [Koopman]. The theory is based on the following three posits. (1) The symmetry principle, according to which, if our relevant knowledge regarding two or more events of the same kind is perfectly symmetric, we must either attach numerical probabilities to none of them, or else to all of them, in which case their values must all be equal. (2) The convention that all probabilities are measured on a scale from zero to unity, the probability of the contrary of an event being given by the formula of complementary probabilities. (3) The axiom of compound probability (in its general form, involving conditional probabilities). From these the author derives the usual formulas of probability, including that of total probability. The author does not seem to attempt to answer the well-known objections to this position. *B. O. Koopman* (New York, N. Y.).

Fiala, F. Essai sur les notions d'ouverture et de fermeture. Dialectica 1, 147–158 (1947).

On a beaucoup insisté ces derniers temps sur la nécessité pour la science et la philosophie de rester ouvertes, c'est-à-dire d'être prêtes à remettre en question leurs principes les mieux établis. L'auteur cherche à montrer, par un certain nombre d'exemples, que la tendance opposée est également légitime et garantit la cohérence de la connaissance. La méthode axiomatique, forme mathématique privilégiée de la connaissance dialectique, satisfait remarquablement à ces deux exigences. *Author's summary.*

Destouches-Février, Paulette. Les notions d'objectivité et de subjectivité en physique atomique. Dialectica 1, 127–146 (1947).

ALGEBRA

Kerawala, S. M. Note on symmetrical incomplete block designs: $\lambda=2$, $k=6$ or 7. Bull. Calcutta Math. Soc. 38, 190–192 (1946).

The author improves methods of tactical enumeration introduced by Q. M. Husain and gives shorter proofs for the existence of exactly three nonisomorphic solutions of the design $v=b=16$, $r=k=6$, $\lambda=2$ and for the nonexistence of the design $v=b=22$, $r=k=7$, $\lambda=2$. Both these facts were first proved by Q. M. Husain [Sankhyā 7, 204–208 (1945); 317–322 (1946); these Rev. 7, 233; 8, 127]. *H. B. Mann* (Columbus, Ohio).

Ionescu, D. V. Sur certains détérminants déduits du détérminant de Vandermonde. Bull. Sci. École Polytech. Timişoara 11, 48–53 (1943).

Williamson, John. Note on Hadamard's determinant theorem. Bull. Amer. Math. Soc. 53, 608–613 (1947).

An $n\times n$ matrix with elements ± 1 and determinant $n^{n/2}$ is an H-matrix. It is known that there exist H-matrices of orders (a) $n=2$, (b) $n=p^h+1\equiv 0 \pmod 4$, p a prime, and for certain other values of n. The author shows that there exists an H-matrix of order $q(q+3)$, where q and $q+4$ are products of factors of types (a) and (b); also of orders $n_1n_2(p^h+1)p^h$ and $n_1n_2m(m+3)$, where $n_1>1$, $n_2>1$ are orders of H-matrices and m, $m+4$ are each of the form p^h+1, p an odd prime. *C. C. MacDuffee.*

Brauer, Alfred. On the characteristic equations of certain matrices. Bull. Amer. Math. Soc. 53, 605–607 (1947).

If A, B and C are square matrices of the same order, with each row and each column of A having zero sum and the ijth element of C being C_i+C_j, then AB and $A(B+C)$ have the same characteristic equation. This theorem is proved by a similarity reduction. *W. Givens* (Knoxville, Tenn.).

Brauer, Alfred. Limits for the characteristic roots of a matrix. II. Duke Math. J. 14, 21–26 (1947).

This is a direct continuation of part I [same J. 13, 387–395 (1946); these Rev. 8, 192]. Let $A=(a_{ij})$ be a square matrix of order n over the complex field and let $P_i=\sum_{j=1}^{n}|a_{ij}|-|a_{ii}|$ and $Q_j=\sum_{i=1}^{n}|a_{ij}|-|a_{jj}|$. The author shows that each characteristic root ω of A lies inside or on at least one of the $n(n-1)/2$ ovals

$$|z-a_{ii}|\,|z-a_{jj}|\leqq P_iP_j, \qquad i\neq j,$$

and inside or on at least one of the ovals

$$|z-a_{ii}|\,|z-a_{jj}|\leqq Q_iQ_j, \qquad i\neq j;$$

that

$$|\omega|\leqq M=\tfrac{1}{2}\max_{1\leqq i<j\leqq n}\{|a_{ii}|+|a_{jj}|+[(|a_{ii}|-|a_{jj}|)^2+4P_iP_j]^{\frac{1}{2}}\}$$

and, when $|a_{ii}a_{jj}|>P_iP_j$, $i,j=1,\cdots,n$, $i\neq j$, that

$$|\omega|\geqq m=\tfrac{1}{2}\min_{1\leqq i<j\leqq n}\{|a_{ii}|+|a_{jj}|-[(|a_{ii}|-|a_{jj}|)^2+4P_iP_j]^{\frac{1}{2}}\}>0.$$

He concludes by proving that each characteristic root of a Hermitian matrix A with positive diagonal elements is positive and greater than or equal to m provided that $|a_{ii}a_{jj}|>P_iP_j$, $i,j=1,\cdots,n$; $i\neq j$. *J. Williamson.*

Bäbler, F. Über einen Satz aus der Theorie der Kristallklassen. Comment. Math. Helv. 20, 65–67 (1947).

An elementary proof of the following theorem of C. Jordan [Bull. Soc. Math. France 3, 103–174 (1875), p. 162]. If A and B are orthogonal matrices, T real, $T^{-1}AT=B$, there

exists a real orthogonal matrix S such that $S^{-1}AS=B$. The dependence of S on T is exhibited. *C. C. MacDuffee.*

Littlewood, D. E. Invariants of systems of quadrics. Proc. London Math. Soc. (2) **49**, 282–306 (1947).

The primitive concomitants of a system of ground forms are classified according to type $\{\mu\}$ [cf. D. E. Littlewood, Philos. Trans. Roy. Soc. London. Ser. A. **239**, 305–365 (1944); these Rev. **6**, 41] and according to class $\{\nu\}$, the partition $\{\nu\}$ determining the simple Frobenius subalgebra of order $(f^{(\nu)})^2$ to which the concomitant belongs. It is proved that the number of linearly independent concomitants of type $\{\mu\}$ and class $\{\nu\}$ which are linear in each of r ground forms each of type $\{\lambda\}$ is $qf^{(\nu)}$, where q is the coefficient of $\{\mu\}$ in the expansion of $\{\lambda\}\otimes\{\nu\}$, $\otimes$ denoting Littlewood's "new multiplication" [cf. loc. cit.]. Further, the number of such concomitants which are of degrees $\tau_1, \cdots, \tau_i$ respectively in i of the ground forms is $qK_{\nu\tau}$, where $K_{\nu\tau}$ is the coefficient of the S-function $\{\nu\}$ in the product $h_{\tau_1}\cdots h_{\tau_i}$ of complete homogeneous symmetric functions.

Applying these results, the author specifies and enumerates the complete sets of invariants of degree less than or equal to 9 in any set of ternary, or of degree less than or equal to 10 in any set of quaternary, quadrics. Comparison with H. W. Turnbull and A. Young [Trans. Cambridge Philos. Soc. **23**, 265–301 (1927)] shows that an invariant of 10 quaternary quadrics obtained by Young does not exist.

D. E. Rutherford (St. Andrews).

Abstract Algebra

* **Jónsson, Bjarni, and Tarski, Alfred. Direct Decompositions of Finite Algebraic Systems.** Notre Dame Mathematical Lectures, no. 5. University of Notre Dame, Notre Dame, Ind., 1947. v+64 pp. $1.25.

A generalization is made of the unique factorization theorem for groups with operators [Wedderburn-Remak-Krull-Schmidt-Ore]. It is shown that a corresponding theorem is valid for any finite algebraic system A having a binary operation ("addition"), and a subalgebra which consists of the single element 0, where $0+x=x+0=x$ for all x. The generalization also contains the reviewer's unique factorization theorem for lattices ["Lattice Theory," Amer. Math. Soc. Colloquium Publ., v. 25, New York, 1940, p. 23; these Rev. 1, 325].

Chapter I is devoted to elementary properties of direct products of subalgebras (as contrasted with direct unions of abstract isomorphism types). "Subtractive" subalgebras of A are defined as subalgebras S such that if $a+s\varepsilon S$ or $s+a\varepsilon S$ [$a\varepsilon A$, $s\varepsilon S$], then $a\varepsilon S$; subtractive subalgebras satisfy the modular law $(B\times C)\cap D=B\times(C\cap D)$. If $A=B\times C$, then B and C are subtractive subalgebras. If $(B\times C)\times D$ exists in A, then $(B\times C)\cap(B\times D)=B$. In chapter II a "central" subalgebra is defined as a subalgebra C such that (i) for any $c\varepsilon C$, the equation $c+x=0$ has a solution, (ii) for any operation f_i of A, including the addition operation $f_a(a,a')=a+a'$, we have

$$f_i(a+c,a'+c',\cdots)=f(a,a',\cdots)+f(c,c',\cdots)$$

for all $a, a', \cdots \varepsilon A$ and $c, c', \cdots \varepsilon C$. It follows that $a+c=c+a$ and that $a+c=a'+c$ implies $a=a'$. Further, any central subalgebra is subtractive and an Abelian group. The concept of central subalgebra is preserved under homomorphisms.

In chapter III, there are proved the fundamental exchange theorems. (E1) Let $B\times C=\prod D_\nu$ in A, where B is finite and directly indecomposable ($\prod$ means direct product). Then for some choice λ of ν there exist subalgebras X and Y of D_λ such that $D_\lambda=X\times Y$ and $B\times C=X\times C=B\times Y\times\prod_{\nu\neq\lambda}D_\nu$. (E2) Let B be a finite subalgebra of A and let $B\times C=\prod D_\nu$ in A. Then there exist subalgebras $D_\nu'\leqq D_\nu$ and $D_\nu''\leqq D_\nu$ such that $B\times C=B\times\prod D_\nu'$ and $B\times C=\prod D_\nu''\times C$. From these exchange theorems, unique factorization and refinement theorems follow by induction. Also, if $B\times C=B'\times C'$, where B and B' are isomorphic, then C and C' are isomorphic. Stronger unique factorization theorems (subalgebras are unique, and not merely centrally isomorphic) hold in centerless algebras. In chapter IV, most of the results of chapter III are extended easily to isomorphism types.

G. Birkhoff (Cambridge, Mass.).

Grayev, M. Isomorphisms of direct decompositions in Dedekind structures. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **11**, 33–46 (1947). (Russian. English summary)

Deux théorèmes sur l'existence de raffinements homologues pour un couple de décompositions directes de l'unité dans une structure complète de Dedekind. [Voir A. G. Kurosh, même Bull. Sér. Math. **7**, 185–202 (1943); **10**, 47–72 (1946); ces Rev. **6**, 145; **8**, 309.] *H. Freudenthal.*

Andrunakievitch, V. Semi-radical and radical rings. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 3–5 (1947).

L'auteur remarque que, dans un anneau A, un élément a est quasi-régulier à droite au sens de S. Perlis [Bull. Amer. Math. Soc. **48**, 128–132 (1942); ces Rev. **3**, 264] lorsque $-a$ admet un inverse à droite pour la loi associative $xoy=x+y-xy$ (qui admet 0 comme élément neutre; l'application $x\to 1-x$ est un isomorphisme de cette loi sur la loi multiplicative dans l'anneau A^* obtenu en adjoignant à A un élément unité 1 par le procédé standard). Il appelle semi-radical à droite un élément a tel que $xoa=yoa$ entraîne $x=y$ (ce qui revient à dire que $xa=x$ entraîne $x=0$), et développe l'analogie entre éléments semi-radicaux à droite et éléments quasi-réguliers à droite, d'une part, éléments non diviseurs de 0 à droite et éléments admettant un inverse à droite (pour la multiplication ordinaire) d'autre part. L'analogue d'un corps est un anneau radical au sens de Jacobson [Amer. J. Math. **67**, 300–320 (1945); ces Rev. **7**, 2]; l'analogue d'un anneau sans diviseur de 0 à droite, c'est-à-dire un anneau dont tout élément est semi-radical à droite, est appelé anneau semi-radical à droite par l'auteur. Ce dernier donne en particulier une condition pour qu'un anneau semi-radical (à droite et à gauche) puisse être plongé dans un anneau radical, une condition pour qu'un anneau semi-radical à droite soit un anneau radical, et étend un théorème de Jacobson [loc. cit.] en montrant que si un anneau semi-radical à droite satisfait à la condition minimale pour les idéaux à droite, il est nilpotent. Les théorèmes sont énoncés sans démonstration. *J. Dieudonné* (Nancy).

Shafarevitch, I. On p-extensions. Rec. Math. [Mat. Sbornik] N.S. **20(62)**, 351–363 (1947). (Russian. English summary)

The author proves that, if the field k is algebraic of degree n over the p-adic rationals, and if k does not contain the pth roots of unity, then the p-extensions of k (i.e., those whose Galois groups are p-groups) are in one-to-one correspondence with those subgroups T, of the free group S with $n+1$ generators, for which the factor group S/T is a p-group. The Galois group of the p-extension is isomorphic to the factor group of the corresponding normal subgroup. The

Please see Errata, p. 1001, Math. Rev. v. 12.

proof depends only on group theory, especially Schreier's theory of subgroups of free groups [Abh. Math. Sem. Hamburgischen Univ. 5, 161–183 (1927)], and the determination of the degrees of a chain of fields $K_0=k, K_1, K_2, \cdots$, each of which is maximal Abelian, of type $(p, p, \cdots)$, over the preceding. This answers the question of what p-groups are realizable as Galois groups over such fields. [O. F. G. Schilling, Trans. Amer. Math. Soc. 47, 440–454 (1940); these Rev. 1, 328, has found all realizable l-groups for all $l\neq p$, but not for $l=p$.]

If k is a function field of one variable with an algebraically closed field of constants, of characteristic p, the same theorem holds for the unramified nondegenerate p-extensions of k, if $n+1$ is replaced by the invariant γ of Hasse and Witt [Monatsh. Math. Phys. 43, 477–492 (1936)]. Witt [J. Reine Angew. Math. 174, 237–245 (1936)] had already determined all realizable p-groups in this case. *G. Whaples.*

Birkhoff, Garrett. Three observations on linear algebra. Univ. Nac. Tucumán. Revista A. 5, 147–151 (1946). (Spanish)

A square matrix is an arithmetic mean of permutations if, and only if, the sum of the elements in each row as well as the sum of those in each column is unity. A linear associative algebra A is defined to be complete provided that if A is an invariant subalgebra of an algebra B, then B is the direct sum of A and another subalgebra. It is proved that A is complete if, and only if, A contains a unity element. Let Γ denote a definition of a linear subspace $C(A)$ for every A of a class of linear (not necessarily associative) algebras A with finite base; suppose, furthermore, that $C(A)$ is invariant under every automorphism of A as a ring and that $C(A)=C(A^*)\cap A$ for every scalar extension A^* of A. Let F denote the field over which A is defined and let F^* denote the extension of F giving A^*. Then $C(A^*)=[C(A)]^*$ if F^* can be obtained from F by a succession of separable or transcendental extensions. *J. L. Dorroh.*

Hochschild, G. On the structure of algebras with nonzero radical. Bull. Amer. Math. Soc. 53, 369–377 (1947).

The author considers algebras B, with radical R, such that B/R is separable (or 0). Then B contains a semisimple subalgebra A such that B as a linear space is the direct sum of A and R. The algebra B is called quasicyclic if R, as a two-sided A-module, can be written as a direct sum of submodules $R_0, R_0^2, \cdots, R_0^n$, where n is the index of nilpotency of R; $R^n\neq 0$, $R^{n+1}=0$. If B_1 with the radical R_1 is a second algebra, then B and B_1 are said to be related if the index of nilpotency is the same for R and R_1 and if $B/R^2\cong B_1/R_1^2$. It is shown that to every algebra B with the given property there belongs a maximal related extension C in the following sense: (1) the algebra C is related to B; (2) there exists a homomorphism of C onto B; (3) if an algebra D is related to C and if δ is a homomorphic mapping of D on C, then δ is an isomorphism. Every algebra C with this property (3) is quasicyclic. If B_1 is an algebra related to B and if C_1 is the maximal related extension of B_1, then C and C_1 are isomorphic. The case where B/R is a central simple algebra is studied in detail. *R. Brauer* (Toronto, Ont.).

Tihomirov, A. A generalization of Malcev's theorem on cleft algebras. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 11, 47–58 (1947). (Russian. English summary)

Let G be an algebra (not necessarily of finite rank) over the fundamental field P. If R is the radical of G, and if G contains a semi-simple subalgebra A so that $G=R+A$, $R\cap A=0$, then G is called a cleft algebra and the above decomposition is called a cleaving of G. By a well-known theorem of Wedderburn, if G is separable and of finite rank over P, then G is a cleft algebra. It has been proved by A. Malcev [C. R. (Doklady) Acad. Sci. URSS (N.S.) 36, 42–45 (1942); these Rev. 4, 130] that in this case any two cleavings $G=R+A=R+A'$ are conjugate to one another, i.e., A can be mapped onto A' by an automorphism of the form $g\to g-gr-r'g+r'gr$, where $g\in G$ and r and r' are certain elements of the radical such that $rr'=r'r=r+r'$. The author generalizes this result by proving the following theorem. If the cleft algebra G possesses a nilpotent radical R, and the factor algebra G/R has finite rank and is separable, then any two cleavings of G are conjugate to one another.

J. Levitzki (Jerusalem).

Albert, A. A. Absolute valued real algebras. Ann. of Math. (2) 48, 495–501 (1947).

A real algebra, i.e., an algebra D over the field R of all real numbers, is called an absolute valued algebra if there is a function $f(a)$ on D to R such that $f(0)=0$, $f(a)>0$ if $a\neq 0$, $f(ab)=f(a)f(b)$, $f(a+b)\leqq f(a)+f(b)$, $f(\alpha a)=|\alpha|f(a)$ for every a and b of D and real α. A real algebra is called a normed algebra if the relation $f(ab)\leqq f(a)f(b)$ rather than the equality is assumed. The author first notes that each absolute valued algebra is a division algebra and then proves that the alternative division algebras are the only absolute valued real algebras with a unity quantity. He remarks that this result is a conjecture of I. Kaplansky, whose suggestions also led to the present proof, which is a considerable modification of the proof the author first obtained. This theorem characterizes the algebra V of all real Cayley numbers as the only absolute valued real algebra with a unity quantity which is not associative, the other alternative division algebras over R being R itself, the field C of all complex numbers and the algebra Q of all real quaternions. The author shows, furthermore, that all absolute valued real algebras without a unity are certain isotopes of Q and V, and supplements this result by giving an example of an absolute valued algebra without a unity. Finally he deals with normed algebras, where one of his results is that every real algebra is a normed algebra. *J. Levitzki.*

Kalisch, G. K. On special Jordan algebras. Trans. Amer. Math. Soc. 61, 482–494 (1947).

Von Jordan [Z. Physik 80, 285–291 (1933)], ferner Jordan, von Neumann und Wigner [Ann. of Math. (2) 35, 29–64 (1934)] sind distributive Algebren betrachtet worden, in denen die Multiplikation kommutativ, aber das Assoziativgesetz durch die schwächere Forderung $(xy)x^2=x(yx^2)$ ersetzt ist. Zum Beispiel bilden die Elemente einer assoziativen Algebra über einem Grundkörper k mit von 2 verschiedener Charakteristik eine Jordanalgebra im Sinne der neuen Multiplikationsregel: $a\cdot b=(ab+ba)/2$. In seiner Dissertation nennt der Verfasser diese Jordanalgebren und alle durch Bildung von linearen, bei Jordanmultiplikation abgeschlossenen Unterräumen entstehenden Jordanalgebren speziell. Er greift die Fälle: (1) $\mathfrak{S}$ normal einfach über k, (2) $\mathfrak{S}_J$ Menge aller Elemente einer normaleinfachen Algebra $\mathfrak{S}$ über k, die bei einem involutorischen Antiautomorphismus J festbleiben, heraus und stellt für sie die zu der von Landherr [Abh. Math. Sem. Hamburgischen Univ. 11, 41–64 (1935); Abh. Math. Sem. Hansischen Univ. 12, 200–241 (1938)] und Jacobson [Ann. of Math. (2) 38, 508–517

(1937); **39**, 181–188 (1938); Duke Math. J. **3**, 544–548 (1938)] entwickelten Theorie für Liealgebren analoge Theorie auf.

Satz 2: Die assoziative Algebra $\mathfrak{S}$ ist genau dann einfach, wenn die zugeordnete Jordanalgebra $\mathfrak{A}$ einfach ist: sie ist genau dann normal einfach, wenn dasselbe für $\mathfrak{A}$ gilt. Bei geeigneter Erweiterung des Grundkörpers k zu einem Oberkörper $\mathfrak{k}$ entsteht aus $\mathfrak{S}$ ein Matrizenring $n$ten Grades über $\mathfrak{k}$. Satz 1: Bezeichnen wir in $\mathfrak{S}_\mathfrak{k}$ die zu a transponierte Matrix mit a', so ist $a^J = ta't^{-1}$ (J lässt sich eindeutig zu einem involutorischen Antiautomorphismus von $\mathfrak{S}_\mathfrak{k}$ erweitern). Entweder ist $t=t'$, $\mathfrak{S}_J : k = n(n+1)/2$ (symmetrischer Fall) oder $t=-t'$, $n=2m$, $\mathfrak{S}_J : k = n(n-1)/2$ (schiefsymmetrischer Fall). Wenn $\mathfrak{S}_J : k > 1$, so ist $\mathfrak{S}$ die assoziative Hülle von $\mathfrak{S}_J$ in $\mathfrak{S}$. Satz 3: Wenn $\mathfrak{S}_J : k > 1$, so ist $\mathfrak{S}_J$ eine einfache Jordanalgebra. Satz 4, 5: Wenn $\mathfrak{S}_J : k > 1$, so lässt sich jeder Isomorphismus zwischen $\mathfrak{S}_J$ und $\mathfrak{T}_K$ über k eindeutig zur einem Isomorphismus zwischen $\mathfrak{S}$ und $\mathfrak{T}$ über k fortsetzen; $\mathfrak{S}_J$ und $\mathfrak{S}_K$ sind genau dann isomorph über k, wenn J und K kogredient sind, d.h., ein x in $\mathfrak{S}$ liegt, sodass stets $a^K = xa^Jx^{-1}$ ist. Satz 6: Die Automorphismengruppe der Jordanalgebra $\mathfrak{S}_J$ ist isomorph zur Faktorgruppe der J-orthogonalen Elementen ($0 \neq uu'ek$) nach den von Null verschiedenen Elementen aus k. Hauptsatz: Wenn eine Jordanangebra $\mathfrak{A}$ über k nach Erweiterung des Grundkörpers zu $\mathfrak{k}$ isomorph zu der Jordanalgebra $\mathfrak{S}_J$, die sich von der normaleinfachen Algebra $\mathfrak{S}$ über $\mathfrak{k}$ herleitet, ist, dann ist bereits $\mathfrak{A}$ über k isomorph zu einer Jordanalgebra $\mathfrak{B}_K$.

H. Zassenhaus (Hamburg).

Schafer, R. D. Concerning automorphisms of non-associative algebras. Bull. Amer. Math. Soc. **53**, 573–583 (1947).

Let A be a nonassociative algebra and L, R, T the associative algebras generated by its left multiplications, right multiplications, and both. Jacobson [Duke Math. J. **3**, 544–548 (1937)] and Albert [Trans. Amer. Math. Soc. **55**, 401–419 (1944); these Rev. **6**, 42] observed that an automorphism of A induces one in L, R or T. The author sharpens these results by determining the set of induced automorphisms. The determination is effected in terms of certain ideals in L, R, T; several further results are proved relating properties of these ideals to properties of A.

I. Kaplansky (Chicago, Ill.).

THEORY OF GROUPS

*Burckhardt, Johann Jakob. **Die Bewegungsgruppen der Kristallographie.** Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, 13. Mineralogisch-Geotechnische Reihe, Band II. Verlag Birkhäuser, Basel, 1947. 186 pp. 24.50 Swiss francs; bound, 29 Swiss francs.

Chapter I contains a rigorous introduction to vectors, coordinate transformations, matrices and lattices. A lattice is defined as a discrete set of points whose set of position vectors is closed under subtraction. (The number of dimensions, i.e., of independent vectors, is denoted by ν.) It follows that a lattice is the set of all transforms of a point by a group of translations. The group of all symmetry operations of the lattice contains this translation group as a self-conjugate Abelian subgroup whose quotient group is isomorphic to the subgroup leaving one lattice point invariant. This finite subgroup is called a holohedry. When $\nu=3$, there are seven kinds of holohedry, one for each of the seven crystal systems. Sometimes as many as four different lattices provide the same holohedry; in fact there are fourteen different lattices altogether. The seven holohedries and their subgroups constitute a set of 32 finite groups called crystal classes. These are enumerated in chapter II. There is an elegant "genealogy" on page 72, showing which classes are subgroups of others. Chapter III employs matrices for enumerating the infinite discrete groups of congruent transformations, first for $\nu=2$ (with excellent diagrams, showing the transforms of a small black scalene triangle) and then for $\nu=3$. These are the 230 space groups of Fedorov, Schoenflies and Barlow. The last twenty pages are concerned with those ν-dimensional space groups whose largest finite subgroups (holohedries) are cyclic, symmetric or alternating, of degree ν or $\nu+1$. *H. S. M. Coxeter* (Toronto, Ont.).

Lombardo-Radice, L. Sugli elementi eccezionali dell'algebra legata a un gruppo di ordine finito in un corpo a caratteristica p. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 170–174 (1947).

If G is a group of finite order g and p a prime number dividing g, the group algebra A of G over a field of characteristic p has a radical $N \neq 0$. In a previous paper [Rend. Sem. Mat. Roma (4) **3**, 239–256 (1939); these Rev. **1**, 258], the author had given necessary and sufficient conditions for an element of A to belong to N. He now shows that his sufficient conditions are not necessary conditions.

R. Brauer (Toronto, Ont.).

Garnir, Henri. Une question de théorie des groupes et son application à un problème de vibrations posé par la chimie théorique. Bull. Soc. Roy. Sci. Liége **15**, 357–382 (1946).

Let $G=\{G_i\}$ be a finite group of order g and let $\sigma : G_i \to \sigma_i$ be a representation of G by matrices of degree r over the field of complex numbers. Suppose that $\Sigma : G_i \to \Sigma_i = \Lambda\sigma_i\Lambda^{-1}$ is a similar representation in fully reduced form (i.e., a direct sum of irreducible representations). In the first part of the paper the author gives a general method of obtaining the matrix Λ, given in advance σ and the first rows of each irreducible representation $D^j : G_i \to D_i^j$, $j=1, 2, \cdots$. The method is based on the orthogonality relations between the elements of the irreducible representations of any finite group. It is proved that if σ and the D^j are in unitary form so is the Λ given by the construction. In the second part of the paper tables are given for the characters and first rows of the irreducible representations for a variety of groups with isomorphic matrix representations of degree two or three. Applications of this theory to the study of vibrations are to follow in a later paper. *R. M. Thrall.*

Souprounenko, D. Primitive solvable groups of substitutions. Rec. Math. [Mat. Sbornik] N.S. **20(62)**, 331–350 (1947). (Russian. English summary)

In part I, the author represents substitutions of degree p^n by polynomial functions defined over $GF[p^n]$. Let $\mathfrak{L}$ be the group of substitutions represented by polynomials of the form $A(\xi) = a_1\xi^{p^{n-1}} + a_2\xi^{p^{n-2}} + \cdots + a_n\xi$, where $\xi, a_1, a_2, \cdots, a_n \epsilon GF[p^n]$. Let $\mathfrak{G}$ be a solvable subgroup of $\mathfrak{L}$ and let $\mathfrak{G}$ be a permutation group for no proper additive subgroup of $GF[p^n]$. Such a subgroup is called primary. Let $\mathfrak{G}$ be a general primary solvable subgroup of $\mathfrak{L}$, where here "general" means "maximal." A maximal Abelian normal sub-

group $\mathfrak{H}$ of $\mathfrak{G}$ induces a direct decomposition of the additive group of $GF[p^n]$ such that, among other things, $\mathfrak{H}$ is a permutation group on each summand; this decomposition leads to a direct decomposition of $\mathfrak{H}$ into cyclic factors of degree p^n-1. Over a certain subfield $GF[p^m]$ the field $GF[p^n]$ has a basis, in terms of which $\mathfrak{H}$ can be represented by diagonal matrices, the diagonal elements of which consist of r repetitions of a set of s elements from $GF[p^m]$; and $mrs=n$.

In part II, the author considers the case $n=q$, a prime, and from the three resulting types of maximal normal Abelian subgroups he determines all types of primitive solvable group of degree p^q. He gives special results for the case $q|(p-1)$ and for the cases $p=2$, 3. To do this, he investigates certain unimodular subgroups of the group $\mathfrak{L}$ associated with $GF[q^2]$. *F. Haimo* (St. Louis, Mo.).

Robinson, G. de B. On the representations of the symmetric group. II. Amer. J. Math. **69**, 286–298 (1947). [For part I cf. Robinson, Amer. J. Math. **60**, 745–760 (1938).] The author considers skew diagrams $[\alpha]-[\beta]$ [A. Young, Proc. London Math. Soc. (2) **37**, 441–495 (1934), p. 454, called them distorted tableaux] obtained by deleting a right diagram (tableau) $[\beta]$ of m letters from a right diagram $[\alpha]$ of l letters, α, β denoting partitions. Such skew diagrams lead to representations of the symmetric group S_{l-m} which are in general reducible. The irreducible components are obtained with the aid of MacMahon's lattice permutations. If f_α, φ_α^β denote the number of standard right diagrams $[\alpha]$ and standard skew diagrams $[\alpha]-[\beta]$, respectively, then φ_α^β is also the degree of the reducible representation $[\alpha]-[\beta]$ and

$$(1)\qquad f_\alpha=\sum_\beta \phi_\alpha^\beta f_\beta;\quad (l!/m!)f_\beta=\sum_\alpha \phi_\alpha^\beta f_\alpha;\quad \phi_\alpha^\beta=\sum_\mu \phi_\alpha^\mu \phi_\mu^\beta,$$

where α, μ, β are partitions of l, r, m, respectively, and where $[\alpha]$ contains $[\mu]$ and $[\mu]$ contains $[\beta]$. The formulae (1) are generalisations of well-known formulae [cf. A. Young, Proc. London Math. Soc. (2) **28**, 255–292 (1928), pp. 261–262]. Skew diagrams also furnish a new proof of a formula given by F. D. Murnaghan and by T. Nakayama [cf. Jap. J. Math. **17**, 165–184 (1940), in particular, pp. 182–183; these Rev. **3**, 195; **4**, 340]. *D. E. Rutherford.*

Shimbireva, H. On the theory of partially ordered groups. Rec. Math. [Mat. Sbornik] N.S. **20(62)**, 145–178 (1947). (Russian. English summary)

L'auteur pose la question sous quelles conditions un groupe peut être doué d'un ordre partiel tel que l'ensemble des éléments plus grand que l'identité engendre le groupe entier. Par exemple, c'est possible pour chaque groupe, dont le facteur-commutateur n'est pas périodique. L'auteur démontre que deux décompositions directes de ces groupes-là possèdent un raffinement commun et donne des conditions pour la possibilité de décompositions sous-directes.

H. Freudenthal (Utrecht).

Gelfand, I., and Neumark, M. Unitary representations of the group of linear transformations of the straight line. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 567–570 (1947).

If R is the group of real linear transformations $y=\alpha x+\beta$, $-\infty<\beta<\infty$, $0<\alpha<\infty$, then it contains the subgroup T of translations $x\to x+\beta$ and the subgroup S of dilations $x\to\alpha x$, both commutative, such that $T\cdot S=R$. The group R possesses the family of one-dimensional representations, which are independent of β, of the group $y\to y+\log\alpha$ on the straight line, and no other almost periodic representations. As for irreducible unitary representations in countably-dimensional Hilbert space the following is the only one. Let H be the Hilbert space of functions $f(x)$ in $L_2(-\infty,\infty)$ whose Plancherel transform vanishes in $-\infty<\alpha<0$, or alternatively in $0<\alpha<\infty$. Then $x\to x+\beta$ corresponds to $f(x)\to f(x+\beta)$ and $x\to\alpha x$ corresponds to $f(x)\to\alpha^{\frac{1}{2}}f(\alpha x)$. The proof is based on the Hellinger-Hahn decomposition of H corresponding to a resolution of the identity; it is applied to the resolution of the identity $E(\lambda)$ occurring in

$$T_\beta f=\int_{-\infty}^{\infty} e^{\lambda\beta}dE(\lambda)f,$$

by Stone's theorem for the subgroup T of R.

S. Bochner (Princeton, N. J.).

***Lauritzen, Svend. En Indledning til en gruppeteoretisk Behandling af de ikke orienterbare Flader [An Introduction to a Grouptheoretical Treatment of the Nonorientable Surfaces].** Thesis, University of Copenhagen, 1942. 264 pp. (Danish)

Non-Abelian groups of linear fractional transformations of the first kind [for definition of kind cf. Lauritzen, Mat. Tidsskr. B **1946**, 92–96 (1946); these Rev. **7**, 469] have been studied by Nielsen [Mitt. Math. Ges. Hamburg **8**, part 2, 82–104 (1940); these Rev. **2**, 213, and an unpublished sequel] and are basic for problems in the topology of orientable surfaces. The author generalizes (the results of Nielsen are subsumed) by considering non-Abelian groups of linear fractional transformations of both first and second kind. Such so-called F-groups are basic for problems in the topology of surfaces which are not necessarily orientable. The principal published theorems of Nielsen [loc. cit.], that a group of the type considered is properly discontinuous and that it is finitely generated if and only if its convex figure is compact mod group, remain valid in the wider connotation. A program, indicated by Nielsen at the end of the cited paper, for constructing F-groups (in terms of generators and relations) stepwise from so-called elementary groups by a geometrical process of forming "free products with united subgroups" is considered in great detail and carried to completion in the case of finitely generated groups. Surfaces as such are not mentioned; characters of the surface (such as genus, number of boundaries, fundamental group) are defined in terms of the given F-group. A minor but puzzling error occurs in the statement of theorem 76.1: certain groups Z_i which are generated by non-Euclidean reflections s_j are stated to be free. Since $s_j^2=1$ this is at variance with the usual definition of free group.

R. H. Fox (Princeton, N. J.).

Baer, Reinhold. Splitting endomorphisms. Trans. Amer. Math. Soc. **61**, 508–516 (1947).

Les endomorphismes s'étudient sur un "loop" L, c'est-à-dire sur une loi de composition (non associative) simplifiable à droite et à gauche, aussi facilement que sur un groupe. Le radical $R(\eta)$ d'un endomorphisme η est l'ensemble des x tels que $x\eta^i=0$ pour au moins un i. (Le "loop" L est écrit additivement, il peut y avoir des opérateurs.) Un endomorphisme est dit clivant si l'on peut trouver C tel que $L=R(\eta)+C$, $R(\eta)\cap C=0$, $C=C\eta$. Le clivage est dit uniforme si tout sous-loop à opérateurs S tel que $S\eta\leqq S$ est clivé. L'auteur donne des conditions suffisantes de clivage uniforme soit pour un endomorphisme particulier dans un

loop général, soit pour une famille d'endomorphismes lorsqu'est satisfaite une condition généralisant celle de chaîne finie. *J. Kuntzmann* (Grenoble).

Baer, Reinhold. Endomorphism rings of operator loops. Trans. Amer. Math. Soc. **61**, 517–529 (1947).

L'auteur étudie pour un "loop" L à opérateurs les endomorphismes qui l'appliquent sur un groupe abélien A. Ils forment un anneau θ. L'auteur étudie pour cet anneau l'existence d'unités à droite et à gauche. Un nilendomorphisme est un endomorphisme η tel que son radical $R(\eta)$ [voir l'analyse ci-dessus] soit L. Cette notion est mise en relation avec celle de radical de θ [N. Jacobson, Amer. J. Math. **67**, 300–320 (1945); ces Rev. **7**, 2]. Un endomorphisme η clive L [pour la définition voir R. Baer, ci-dessus] si et seulement si il existe des endomorphismes ϵ et κ dans θ tels que $\eta\kappa=\kappa\eta=\epsilon$, $\kappa\epsilon=\epsilon\kappa=\kappa$, $L=R(\eta-\epsilon\eta)$. L'auteur étudie enfin le cas où tout endomorphisme de θ clive L, ce qui est une forme affaiblie de la condition de chaîne.

J. Kuntzmann (Grenoble).

NUMBER THEORY

Delfeld, Albert. Table des solutions de la congruence $X^4+1\equiv 0 \pmod p$ pour $300000<p<350000$. Inst. Grand-Ducal Luxembourg. Sect. Sci. Nat. Phys. Math. Arch. N.S. **16**, 65–70 (1946).

This table gives the two absolutely least solutions of the congruence in question for the 980 primes $p=8n+1$ between the limits indicated in the title. It is an extension of a previous table of Gloden [Mathematica, Timişoara **21**, 45–65 (1945); these Rev. **7**, 145]. *D. H. Lehmer.*

Gloden, A. Factorisation de nombres de la forme $16x^8+1$. Euclides, Madrid **7**, 95 (1947).

This note gives factors of $16x^8+1$ for certain integers x between 23 and 100. There are in all 33 complete factorizations. In seven cases only one or two small factors are given, the residual factors being greater than 500000. Since these numbers are of the form y^4+1, use was made of recent tables of solutions of the congruence $y^4+1\equiv 0 \pmod p$ [see the preceding review]. *D. H. Lehmer* (Berkeley, Calif.).

Jarden, Dov. Table of the ranks of apparition in Fibonacci's sequence. Riveon Lematematika **1**, 54 (1946). (Hebrew)

Let $U_0=0$, $U_1=1$, $U_{n+1}=U_n+U_{n-1}$, be the Fibonacci sequence. The author denotes by $a(p)$ the rank of apparition of the prime p, that is, the least positive subscript n for which U_n is divisible by p. The table gives for each prime $p\leqq 1511$ the value of $a(p)$ and the factorization of $p-\left(\frac{5}{p}\right)$. Those factors whose product is the "residue index" $\{p-\left(\frac{5}{p}\right)\}/a(p)$ are underlined. *D. H. Lehmer.*

Klee, V. L., Jr. Some remarks on Euler's totient. Amer. Math. Monthly **54**, 332 (1947).

Record of solutions $n<3000$ of $\varphi(n)=\varphi(n+1)$, $\varphi(n)=\varphi(n+2)$, $\varphi(2n-1)<\varphi(2n)$, etc., found from J. W. L. Glaisher's "Number-Divisor Tables" [Cambridge University Press, 1940; these Rev. **2**, 33]. *N. G. W. H. Beeger.*

Goodstein, R. L., and Rumney, M. Some new theorems on divisibility. Math. Gaz. **31**, 90–92 (1947).

Let $m\geqq 2$ be a fixed integer; then any integer $N=ma+b$, $b<m$. The $T^m_{p,q}$ transform of N is defined to be $pa+qb$. The main result of the paper is as follows. Assume $(a, b)=1$, $(p, q)=1$, $T^m_{p,q}F=F$. Write $t_0=N$, $t_{r+1}=T^m_{p,q}t_r$. If for some k, $t_{k+1}>t_k$, then F does not divide N. If $t_{r+1}<t_r$, $0\leqq r\leqq\lambda-1$, and $t_{\lambda+1}=t_\lambda$, then $F\mid N$. *L. Carlitz* (Durham, N. C.).

Ranga Chariar, V. On certain sequences of integers no one of which is divisible by any other. Patna Univ. J. **1**, 22–29 (1944).

The sequence $B(b_1)$, $b_1<\cdots<b_k<\cdots$ is defined as follows: b_k occurs in it if and only if it is not a multiple of $b_1, \cdots, b_{k-1}$. The author proves among other things that $B(b_1)$ coincides with the sequence of primes after p^2, where p is the greatest prime less than b_1.

Let $a_1<a_2<\cdots$ be a sequence of integers no one of which divides the other. The reviewer proved that $\sum 1/(a_k \log a_k)$ converges. The author raises the question of whether $\sum 1/\{a_k(\log a_k)^\delta\}$, $\delta<1$, also converges. [It can be shown that this is not so, in fact that if $f(k)$ is any function tending to infinity, there exists a sequence $a_1<\cdots<a_k<\cdots$, no element of which divides another, such that $\sum f(k)/(a_k \log a_k)$ diverges.] *P. Erdös* (Syracuse, N. Y.).

Prasad, Sarveshwar, and Chariar, V. R. On certain Diophantine equations. Patna Univ. J. **2**, 66–71 (1946).

L. J. Mordell [J. London Math. Soc. **17**, 199–203 (1942); these Rev. **4**, 265; **6**, 334] has discussed certain Diophantine equations having an infinite number of solutions; in particular, a set of solutions was exhibited for $z^2=x^3+y^3-1$. The present paper exhibits other infinite sets of solutions of this equation and also of the equation $2z^2=x^3+y^3-1$. The method used is similar to Mordell's. *I. Niven.*

Moessner, Alfred. Einige numerische Identitäten. Bull. Sci. École Polytech. Timişoara **9**, 245–255 (1940).

The paper also appeared in Proc. Indian Acad. Sci., Sect. A. **10**, 296–306 (1939); these Rev. **1**, 133.

Pizá, Pedro A. Cubic and quintic triangles. Revista Soc. Cubana Ci. Fís. Mat. **2**, 92–97 (1946). (Spanish)

Formulas in terms of parameters are derived for the sides a, b, c of a Fermat triangle of orders 3 and 5, i.e., one whose sides satisfy the equation $a^n+b^n=c^n$ $(n=3, 5)$. These formulas involve square roots only. A ruler and compass construction is given for the case $n=5$. *H. W. Brinkmann.*

Pizà, Pedro A. Construction euclidienne de triangles de Fermat. Mathesis **55**, 5–8 (1947).

Two constructions, by ruler and compass, are given for a Fermat triangle of order 3, i.e., a triangle whose sides a, b, c satisfy the equation $a^3+b^3=c^3$. *H. W. Brinkmann.*

Ljunggren, Wilhelm. Sur une généralisation d'un théorème de C. Størmer. Arch. Math. Naturvid. **48**, no. 11, 145–152 (1944).

C. Størmer [Skr. Vid.-Selsk. Christiania. I. **1895**, no. 11] found all solutions of $m \arctan(1/x)+n \arctan(1/y)=k\pi/4$ in integers m, n, x, y, k. The author solves the same problem for $m \arctan(1/x\sqrt{D})+n \arctan(1/y\sqrt{D})=k\pi/2$, where $D>1$ is square-free, there being 8 nontrivial solutions in integers m, n, x, y, k, D. The author [Ark. Mat. Astr. Fys. **29A**, no. 13 (1943); these Rev. **6**, 265] had previously solved the more general equation

$$m \arctan(\sqrt{D}/x)+n \arctan(\sqrt{D}/y)=k\pi/2$$

for the cases $D=2$ and $D=3$. *I. Niven* (Eugene, Ore.).

Châtelet, F. Sur l'arithmétique des courbes de genre un. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **22**, 153–165 (1946).

A curve C of genus one with rational coefficients is transformable by a birational transformation $\mathfrak{R}$ into a Weierstrass cubic W of the form $y^2=x^3+Ax+B$. It is shown that W may be chosen to have rational coefficients, the coefficients of $\mathfrak{R}$ being in a normal algebraic extension k of the rational ground-field. The existence of a rational point on C is equivalent to the existence on W of a rational point satisfying certain conditions depending on the Galois group of k.

Mordell [Proc. Cambridge Philos. Soc. **21**, 179–192 (1922)] and Weil [Bull. Sci. Math. (2) **54**, 182–191 (1930)] have demonstrated the existence of a "base" for rational points on W, but it is not known how to determine such a base. If a base for W is given then a finite number of operations will determine whether or not C contains a rational point. *D. B. Scott* (Aberdeen).

Châtelet, François. Méthode galoisienne et courbes de genre un. Ann. Univ. Lyon. Sect. A. (3) **9**, 40–49 (1946).

Generalisation of the results of the paper reviewed above to ground fields other than the field of rational numbers. *D. B. Scott* (Aberdeen).

Davenport, H. Non-homogeneous binary quadratic forms. II. Nederl. Akad. Wetensch., Proc. **50**, 378–389 = Indagationes Math. **9**, 236–247 (1947).

[For part I see the same Proc. **49**, 815–821 = Indagationes Math. **8**, 518–524 (1946); these Rev. **8**, 444.] The author investigates the particular case $\xi=x+\theta y$, $\xi'=x+\theta' y$, where $\theta=\frac{1}{2}(1+5^{\frac{1}{2}})$, $\theta'=\frac{1}{2}(1-5^{\frac{1}{2}})$, and asks whether there are integers x, y such that (1) $|(\xi-a)(\xi'-b)|\leqq M$. Theorem 1. The inequality (1) always holds for $M=\frac{1}{4}$; the sign of equality is needed if and only if (2) $a=\frac{1}{2}\tau+\xi_1$, $b=\frac{1}{2}\tau'+\xi_1'$, where τ is any unit and ξ_1 is any integer of the field $k(\theta)$ and τ', ξ_1' their conjugates. Theorem 2. If a and b are not of the form (2), then (1) holds for $M=\frac{1}{5}$; the sign of equality is needed if and only if (3) $a=5^{-\frac{1}{2}}\tau+\xi_1$, $b=-5^{-\frac{1}{2}}\tau'+\xi_1'$. Theorem 3. If a and b are not of the form (3), nor of the form (2), then (1) holds for $M=1/6.34$, where the number 6.34 could be improved slightly. Theorem 4 gives an answer to the question of whether there are infinitely many integers x, y such that (1) holds: if $M>\frac{1}{4}$, (1) has infinitely many solutions with $|\xi-a|$ arbitrarily small and also infinitely many solutions with $|\xi'-b|$ arbitrarily small. *J. F. Koksma.*

Davenport, H. The product of n homogeneous linear forms. Nederl. Akad. Wetensch., Proc. **49**, 822–828 = Indagationes Math. **8**, 525–531 (1946).

Let $L_1, \cdots, L_n$ be n homogeneous linear forms in n variables $u_1, \cdots, u_n$ with real coefficients and determinant 1. Then, improving results of Minkowski and Blichfeldt, the author proves that the lower bound M of $|L_1 \cdots L_n|^{1/n}$ for integral values of $u_1, \cdots, u_n$ satisfies $M\leqq(n^{-1}\gamma_n k^{-1+1/n})^{\frac{1}{2}}$, where $k>1.4$ and $\gamma_n\leqq 2\pi^{-1}\{\Gamma(2+\frac{1}{2}n)\}^{n/2}$. [For literature see, for example, the reviewer's "Diophantische Approximationen," Ergebnisse der Math., vol. 4, no. 4, Springer, Berlin, 1936.] *J. F. Koksma* (Amsterdam).

Davenport, H., and Heilbronn, H. On indefinite quadratic forms in five variables. J. London Math. Soc. **21**, 185–193 (1946).

It has been conjectured that any indefinite quadratic form in 5 variables with real coefficients assumes values that are arbitrarily small numerically (for integer values of the coefficients). The authors prove that this is the case when the form $Q(x_1, \cdots, x_5)=\lambda_1 x_1^2+\cdots+\lambda_5 x_5^2$. The corresponding problem for 3 and 4 variables has not been solved yet; the authors assert that their method, which is a modification of the Hardy-Littlewood method, also holds for the general case $Q(x_1, \cdots, x_s)=\lambda_1 x_1^k+\cdots+\lambda_s x_s^k$, where $s=2^k+1$. Their exact result is as follows. Let $\lambda_1, \cdots, \lambda_5$ be real numbers, none of them zero, such that at least one of the ratios λ_r/λ_s is irrational. Then there exist arbitrarily large integers P, such that the inequalities $1\leqq x_1\leqq P, \cdots, 1\leqq x_5\leqq P$, $|Q(x_1, \cdots, x_5)|<1$ have more than γP^3 solutions, where $\gamma=\gamma(\lambda_1, \cdots, \lambda_5)>0$. *J. F. Koksma* (Amsterdam).

Estermann, T. Note on a theorem of Minkowski. J. London Math. Soc. **21**, 179–182 (1946).

The author gives a new simple proof of the following theorem. Let $0<\lambda_1\leqq\cdots\leqq\lambda_n$ and let S be a convex set of points in n-dimensional space with the following property: for any integer k from 1 to n and any two points $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ of $\lambda_k S$, where $x_1-y_1, \cdots, x_n-y_n$ are integers, we have $x_k=y_k$, $x_{k+1}=y_{k+1}, \cdots, x_n=y_n$. Then $\lambda_1\lambda_2\cdots\lambda_n V(S)\leqq 1$. Here $V(S)$ is the n-dimensional volume of S and λS denotes the set of the points $(\lambda x_1, \cdots, \lambda x_n)$ corresponding to points $(x_1, \cdots, x_n)$ of S. *V. Knichal.*

Jarník, Vojtěch, and Knichal, Vladimír. On the main theorem of the geometry of numbers. Rospravy II. Třídy České Akad. **53**, no. 43, 15 pp. (1943). (Czech)

The authors consider the extension of Minkowski's theorem to an arbitrary set A in r-dimensional space. Let $J(A)$ be the inner Lebesgue measure of A and let $\mathfrak{B}(A)$ be the set of all points $P-Q$ for $P\in A$, $Q\in A$, the subtraction being defined in an obvious way. Further, let $\tau_j''(N)$ denote the lower bound of all numbers $\alpha>0$ for which the set $\sum_{0<\beta\leqq\alpha}\beta N$ contains at least j independent lattice points. Here βN is the set obtained from an arbitrary set N by expansion about the origin in the ratio β. The most important part of the authors' first theorem is that, if $0<J(A)<\infty$ and $\tau_1''(\frac{1}{2}\mathfrak{B}(A))\neq 0$, then $\mu_1\mu_2\cdots\mu_r J(A)\leqq 2^r$, where $\mu_1, \mu_2, \cdots, \mu_r$ are any numbers such that $0<\mu\leqq\tau_j''(\frac{1}{2}\mathfrak{B}(A))$ $(1\leqq j\leqq r)$ and $\mu_2/\mu_1, \cdots, \mu_r/\mu_{r-1}$ are integers. From this it follows that

$$T(A)=\tau_1''(\tfrac{1}{2}\mathfrak{B}(A))\tau_2''(\tfrac{1}{2}\mathfrak{B}(A))\cdots\tau_r''(\tfrac{1}{2}\mathfrak{B}(A))\leqq 2^{2r-1},$$

and it is shown that the upper bound 2^{2r-1} can be improved slightly. For a bounded convex set A, $T(A)\leqq 2^r$ by Minkowski's theorem. However, 2^r is not the correct upper bound for an arbitrary set A as is shown by the authors' second theorem where an example is given of a set A for which $T(A)>2^r$. The main theorem is an extension of a result due to Jarník [Věstník Královské České Společnosti Nauk. Třida Matemat.-Přírodověd., **1941**; these Rev. **7**, 417] who proved a less general result for a bounded closed set A. Use is made in the proof of a generalisation of a theorem of Blichfeldt due to Jessen [cf. Fenchel, Acta Arith. **2**, 230–241 (1937)]. [Results of a similar but less general type have been given independently by Rogers in the paper reviewed below.] *R. A. Rankin.*

Rogers, C. A. A note on a theorem of Blichfeldt. Nederl. Akad. Wetensch., Proc. **49**, 930–935 = Indagationes Math. **8**, 589–594 (1946).

Let L be an n-dimensional lattice with determinant $\Delta>0$. Let S be a closed bounded set with inner volume $V(S)>0$. If $\lambda>0$, we denote by λS the set of all points λx, where $x\in S$. We define $2n$ numbers $0<\lambda_1\leqq\cdots\leqq\lambda_n$, $0<\mu_1\leqq\cdots\leqq\mu_n$

($\lambda_k \leqq \mu_k$) in the following way: λ_1 is the smallest number for which there exists a nonvanishing lattice-vector A_1 which is contained in $\lambda_1 S$ (i.e., $A_1 = \overline{xy}$, where $x \varepsilon \lambda_1 S$, $y \varepsilon \lambda_1 S$); λ_2 is the smallest number for which $\lambda_2 S$ contains a lattice-vector A_2, linearly independent of A_1, etc. (e.g., $\lambda_3 S$ contains a lattice-vector A_3, linearly independent of A_1, A_2); μ_k is the smallest number such that the set μS contains, for every $\mu \geqq \mu_k$, at least k linearly independent lattice-vectors. If S is convex and symmetrical about the origin, we have $\lambda_k = \mu_k$, $\lambda_1 \lambda_2 \cdots \lambda_n V(S) \leqq \Delta$ [Minkowski]. Results. (I) We have $(\lambda_2^n/k) V(S) \leqq \Delta$, $(\mu_2^n/k') V(S) \leqq \Delta$, where k, k' are integers, $\lambda_2/\lambda_1 \leqq k < \lambda_2/\lambda_1 + 1$, $\mu_2/\lambda_1 \leqq k' < \mu_2/\lambda_1 + 1$. (II) If $n = 2$ then to every $\epsilon > 0$ there corresponds an S such that $\lambda_1 \lambda_2 V(S) > (\frac{4}{9} - \epsilon)\Delta$. [For more general results (except that concerning μ_2) see the paper of Jarník and Knichal reviewed above.] *V. Jarník* (Prague).

Mahler, K. Lattice points in n-dimensional star bodies. Univ. Nac. Tucumán. Revista A. **5**, 113–124 (1946).

This is a report of a lecture delivered in May, 1945. It contains an informal account of work done by the author recently in n-dimensional lattice point theory. The results described have, in the meantime, appeared in two papers [Proc. Roy. Soc. London. Ser. A. **187**, 151–187 (1946); Nederl. Akad. Wetensch., Proc. **49**, 331–343, 444–454, 524–532, 622–631 (1946); these Rev. **8**, 195, 12]. The descriptions are illustrated by special examples. No complete proofs are given. *D. Derry* (Vancouver, B. C.).

Robinson, Raphael M. Unsymmetrical approximation of irrational numbers. Bull. Amer. Math. Soc. **53**, 351–361 (1947).

By means of geometrical methods B. Segre [Duke Math. J. **12**, 337–365 (1945); these Rev. **6**, 258] deduced a theorem on asymmetric approximation and C. D. Olds [same Bull. **52**, 261–263 (1946); these Rev. **8**, 196] gave a proof making use of Farey series. The author makes use of continued fractions and proves, among other results, that for any ξ and any $\epsilon > 0$ the inequality

$$-\frac{1}{(5^{\frac{1}{2}} - \epsilon)B^2} < \frac{A}{B} - \xi < \frac{1}{(5^{\frac{1}{2}} + 1)B^2}$$

has infinitely many integer solutions. *J. F. Koksma.*

van der Corput, J. G. On sets of integers. I, II. Nederl. Akad. Wetensch., Proc. **50**, 252–261, 340–350 = Indagationes Math. **9**, 159–168, 198–208 (1947).

The first two paragraphs of this paper contain modifications of proofs and improvements of theorems proved by the reviewer [Ann. of Math. (2) **43**, 523–527 (1942); these Rev. **4**, 35] and by F. J. Dyson [J. London Math. Soc. **20**, 8–14 (1945); these Rev. **7**, 365]. In the third paragraph weight functions $f(m)$ are considered satisfying $f(m) > 0$, $f(m+1) \geqq f(m)$, $f^2(m+1) \geqq f(m) f(m+2)$ and the following theorem established. Let A, B be sets of nonnegative integers both containing the number 0, $A+B$ their sum. Let, furthermore, $A(m)$ denote the number of positive integers in A not exceeding m. If $A(m) + B(m) \geqq \gamma \sum_1^m f(h)$, $h = 1, \cdots, g$, $\gamma \leqq 1$, then $(A+B)(m) \geqq \gamma \sum_1^m f(h)$, $h = 1, \cdots, g$. *H. B. Mann* (Columbus, Ohio).

Selberg, Sigmund. Note on a metrical problem in the additive theory of numbers. Arch. Math. Naturvid. **48**, no. 8, 111–118 (1944).

Let A, B be sets of nonnegative integers; $A \supset 0$, $B \supset 0$, 1. Let $l(m)$ denote the least number of elements in B required to write A as a sum of b's. Put

$$\lambda = \operatorname*{fin\,sup}_{n=1,2,\cdots} \sum_{m=1}^{n} l(m)/n.$$

Let α be the density of A, γ that of $A+B$. The author proves that $\gamma > \alpha(1 + 3(1-\alpha)/4\lambda)$. This is an improvement over a previous result by Landau [Über einige neuere Fortschritte der additiven Zahlentheorie, Cambridge University Press, 1937, pp. 3–6, 60–63]. For $1 > \alpha > \frac{1}{3}$ this is also better than the estimate $\gamma > \alpha(1 + (1-\alpha^{\frac{1}{2}})/\lambda)$ given by A. Brauer [Math. Z. **44**, 212–232 (1938)]. The author's methods are modifications of Landau's. *H. B. Mann* (Columbus, Ohio).

Hua, Loo-Keng. Some results on additive theory of numbers. Proc. Nat. Acad. Sci. U. S. A. **33**, 136–137 (1947).

Several statements are made without proof, the most important being that Hardy and Littlewood's asymptotic formula for the number of representations of a number as the sum of s kth powers holds for

$$s \geqq 4k^2(\log k + \tfrac{1}{2}(\log k^2)^{\frac{1}{2}} + \tfrac{1}{4} \log \log k + 1).$$

T. Estermann (London).

Gupta, Hansraj. A table of values of $N_3(t)$. Proc. Nat. Inst. Sci. India **13**, 35–63 (1947).

The main table of this paper gives the number $n_3(k)$ of sets (x, y, z) of nonnegative integers for which $x^2 + y^2 + z^2 = k$ for $k = 1(1)10000$. These values are subtotalled at intervals of 8, thus giving values of the sum function $N_3(t) = \sum_{k=0}^{t} n_3(k)$ for $t = 8m - 1$, $m = 1(1)1250$. There is a short table, based on the main table, giving data on the lattice points inside circles and spheres. Let $C(r)$ and $S(r)$ denote the number of points with integer coordinates inside or on the circle and sphere, respectively, of radius r about the origin. The table gives these functions for $r = 1(1)100$ together with $N_3(r^2)$ and $N_2(r^2)$ (the number of pairs (x, y) of nonnegative integers for which $x^2 + y^2 \leqq r^2$). For comparison with $S(r)$, the nearest integer to the volume of a sphere of radius r is also given. *D. H. Lehmer* (Berkeley, Calif.).

Gupta, Hansraj. A note on the parity of $p(n)$. J. Indian Math. Soc. (N.S.) **10**, 32–33 (1946).

A short proof is given of the congruence of MacMahon, $p(n) \equiv \sum p(t) \pmod 2$, where $p(m)$ denotes the number of unrestricted partitions of m, $p(0) = 1$, and t ranges over all nonnegative integers of the form $(2n - i^2 - i)/8$, $i = 0, 1, \cdots$. The proof makes use of the fact that Ramanujan's function $\tau(n)$ is odd if and only if n is an odd square. *D. H. Lehmer* (Berkeley, Calif.).

Motzkin, Theodor. Ordered and cyclic partitions. Riveon Lematematika **1**, 61–67 (1947). (Hebrew)

A partition is the representation of a positive integer as a sum of positive integers (elements). If two partitions in which the same elements appear in different order are counted different, we have ordered partitions; if two partitions in which the same elements appear in different order are counted different except when they originate from each other by a cyclic permutation of the elements (when they are regarded as identical), we have cyclic partitions. The inverse of a (cyclic or ordered) partition is the partition which consists of the same elements in reversed order. If a partition and its inverse are considered as identical, we have directionless partitions. The author determines the number of ordered and cyclic partitions of a given positive integer n, and investigates restricted partitions whose ele-

ments are taken from a given set of positive integers; he also discusses directionless partitions. *A. Erdélyi.*

Whiteman, Albert Leon. A sum connected with the partition function. Bull. Amer. Math. Soc. **53**, 598–603 (1947).

The $A_k(n)$ appearing in the formula for the partition function $p(n)$ have been evaluated by D. H. Lehmer [Trans. Amer. Math. Soc. **43**, 271–295 (1938)]. Other proofs of the relevant factorization formulas have been given by Rademacher and Whiteman [Amer. J. Math. **63**, 377–407 (1941); these Rev. **2**, 249]. New proofs of the evaluation of the $A_k(n)$ when k is a prime power are given in the present paper. *H. S. Zuckerman* (Seattle, Wash.).

Lahiri, D. B. On Ramanujan's function $\tau(n)$ and the divisor function $\sigma_k(n)$. I. Bull. Calcutta Math. Soc. **38**, 193–206 (1946).

This first paper of a series has to do only with the function $n^r\sigma_k(n)$, where σ is the sum of the kth powers of the divisors of n. It develops in a systematic way no fewer than 89 different congruences connecting values of this function for various values of r and k but with n fixed. We have, for example, $\sigma_{13}(n)\equiv 380\sigma_7(n)-(840n-441)\sigma_5(n)+20\sigma_1(n)$ (mod 423360). The method is a development of a scheme of Ramanujan [Collected Papers, nos. 18 and 28] based on his function $\Phi_{r,s}(x)=\sum_{m,n=1}^{\infty}m^r n^s x^{mn}$ and the fact that this function is only a polynomial in the invariants $P=1-24\sum\sigma_1(n)x^n$, $Q=1+240\sum\sigma_3(n)x^n$, $R=1-540\sum\sigma_5(n)x^n$. The author finds by inversion all products of two or more Φ's which can be expressed linearly in terms of the Φ's. Identifying coefficients in various ways, he obtains the desired congruences. *D. H. Lehmer* (Berkeley, Calif.).

Venkataraman, C. S. On some remarkable types of multiplicative functions. J. Indian Math. Soc. (N.S.) **10**, 1–12 (1946).

The author considers multiplicative functions of several arguments of the types called "ordinal" and "modular" by Vaidyanathaswamy [Trans. Amer. Math. Soc. **33**, 579–662 (1931)]. He proves a number of theorems concerning the properties of such functions when certain of the arguments are subjected to the process of "convolution." *R. A. Rankin* (Cambridge, England).

Mills, W. H. A prime-representing function. Bull. Amer. Math. Soc. **53**, 604 (1947).

A prime-representing function is one which is a prime number for every integral value of the variable. A. E. Ingham [Quart. J. Math., Oxford. Ser. **8**, 255–266 (1937)] proved that $p_{n+1}-p_n<Kp_n^{5/8}$, where p_n denotes the nth prime and K is a fixed positive integer. By means of this the author proves that there exists a real number A such that $[A^{3^n}]$ is a prime-representing function. Inconvenient misprints: in (3), (4), (5) $3-n$ should be 3^{-n}; $3-n-1$ should be 3^{-n-1}. *N. G. W. H. Beeger* (Amsterdam).

Selberg, Atle. Contributions to the theory of the Riemann zeta-function. Arch. Math. Naturvid. **48**, no. 5, 89–155 (1946).

It is first proved that, if $N(\sigma, T)$ is the number of zeros $\beta+i\gamma$ of $\zeta(s)$ with $\beta>\sigma$ and $0<\gamma<T$, then $N(\sigma, T)=O(T^{2-2(\sigma-\frac{1}{2})}\log T)$ uniformly for $\frac{1}{2}\leqq\sigma\leqq 1$. For values of σ near to $\frac{1}{2}$ this is an improvement on a result of Ingham, which contained a factor $\log^5 T$. The remainder of the paper is devoted to the study of $S(t)=\pi^{-1}\arg\zeta(\frac{1}{2}+it)$ and $S_1(T)=\int_0^t S(t)dt$. The most remarkable results are

$$S(t)=\Omega_{\pm}\{(\log t)^{\frac{1}{2}}(\log\log t)^{-\frac{1}{2}}\},$$
$$S_1(t)=\Omega_{\pm}\{(\log t)^{\frac{1}{2}}(\log\log t)^{-10/3}\},$$
$$S_1(t)=\Omega_{+}\{(\log t)^{\frac{1}{2}}(\log\log t)^{-\frac{1}{2}}\}.$$

The last result approaches what was previously known only on the assumption of the Riemann hypothesis. *E. C. Titchmarsh* (Oxford).

Titchmarsh, E. C. On the zeros of the Riemann zeta function. Quart. J. Math., Oxford Ser. **18**, 4–16 (1947).

The author gives a shorter proof for Selberg's inequality $N_0(T)>AT\log T$ [Skr. Norske Vid. Akad. Oslo. I. **1942**, no. 10; these Rev. **6**, 58] for the zeros of the zeta function in the interval $(0, T)$ of the critical line. "This is done by using a Fourier transform method similar to that which I used in my Cambridge tract to prove" $N_0(T)>AT$. *N. Levinson* (Cambridge, Mass.).

Wang, F. T. A mean-value theorem of the Riemann zeta function. Quart. J. Math., Oxford Ser. **18**, 1–3 (1947).

In this note the author states that

$$T^{-1}\int_0^T|\zeta(\sigma+it)|^2\log|\zeta(\sigma+it)|dt\sim\zeta(2\sigma)\log\zeta(2\sigma),\quad \sigma>\tfrac{1}{2}.$$

For the case $\sigma=\frac{1}{2}$, the author is not able to obtain by his method an asymptotic result, analogous to

$$\int_0^T|\zeta(\tfrac{1}{2}+it)|^2dt\sim T\log T$$

[Titchmarsh, The Zeta-Function of Riemann, Cambridge University Press, 1930, p. 34]. He only states without proof the order-result

$$\int_0^T|\zeta(\tfrac{1}{2}+it)|^2\log|\zeta(\tfrac{1}{2}+it)|dt=O(T\log T\log\log T).$$

Moreover he states without proof the generalization of the first result,

$$\int_0^T|\zeta(\sigma+it)|^{2k}\log|\zeta(\sigma+it)|dt\sim TC_{2k}(2\sigma)\sum_p\frac{\sum_{\nu=1}^{\infty}\binom{\nu+k-1}{\nu}E(\nu)p^{-2\nu\sigma}}{\sum_{\nu=0}^{\infty}\binom{\nu+k-1}{\nu}^2p^{-2\nu\sigma}}$$

for $0<k\leqq 2$, $\sigma>\frac{1}{2}$; here

$$C_k(\sigma)=\prod_p\left\{\sum_{\nu=0}^{\infty}\binom{\nu+k-1}{\nu}^2p^{-2\sigma\nu}\right\},\quad E(\nu)=\sum_{\mu=1}^{\nu}\frac{\mu+k-1}{\mu^2}.$$

[Reviewer's remark: The last result seems to be inconsistent with the first for $k=1$. Should not $C_{2k}(2\sigma)$ in the second member be $C_k(\sigma)$?] *S. C. van Veen* (Delft).

Brauer, Richard. On the zeta-functions of algebraic number fields. Amer. J. Math. **69**, 243–250 (1947).

L'auteur démontre, d'abord, le résultat que la fonction $\zeta(s, K)$ d'une extension galoisienne finie K d'un corps de nombres algébriques de degré fini k se divise par $\zeta(s, k)$, autrement dit que $\zeta(s, K)/\zeta(s, k)$ est une fonction entière de s. [Le résultat analogue a déjà été prouvé par Artin pour les extensions K/k dont le groupe de Galois est le groupe d'icosaèdre.] Celà résulte du lemme suivant de la théorie des groupes: soient $\mathfrak{G}$ un groupe d'un ordre fini g, θ le caractère de sa représentation régulière, χ_0 son 1-caractère; alors $\theta-\chi_0$ peut être représenté comme une combinaison linéaire $\sum c_\rho\omega_\rho^*$ de caractères ω_ρ^* induits dans $\mathfrak{G}$ par les

caractères ω_ρ de ses sous-groupes cycliques A, autres que les 1-caractères de ces sous-groupes, et où les c_ρ sont des nombres rationnels non-négatifs de dénominateur g. Pour prouver ce lemme, l'auteur forme, pour tout sous-groupe cyclique A de $\mathfrak{G}$, une combinaison linéaire $\psi_A(\alpha)$ ($\alpha \varepsilon A$) de caractères de A telle que, a étant l'ordre de A et $\varphi(x)$ étant l'indicatrice d'Euler, $\psi_A(\alpha)$ soit $a\varphi(a)$, 0 ou $-a$ suivant que $\alpha=1$, $\alpha \neq 1$ et n'engendre pas A ou α engendre A. Il se trouve que les coefficients des caractères irréductibles de A dans $\psi_A(\alpha)$ sont des entiers non-négatifs dont celui du 1-caractère de A est nul, et que, si $\psi_A^*(\sigma)$ ($\sigma \varepsilon \mathfrak{G}$) est le caractère induit par $\psi_A(\alpha)$ dans $\mathfrak{G}$, la somme $\sum_A \psi_A^*(\sigma)$, étendue à tous les sous-groupes cycliques d'ordre supérieur à 1 de $\mathfrak{G}$, est précisément $g(\theta-\chi_0)$, ce qui prouve le lemme. Il résulte du lemma que $[\zeta(s, K)/\zeta(s, k)]^g = L(s; \theta-\chi_0, K/k)^g$, où on a pris comme $\mathfrak{G}$ le groupe de Galois de K/k, est une fonction entière, et, puisque il est connu que $\zeta(s, K)/\zeta(s, k)$ est méromorphe, cette dernière fonction est aussi entière.

Ensuite, l'auteur applique ce résultat pour démontrer que, pour les corps de nombres algébriques k d'un degré fixe n, $\log(hR)$, où h est le nombre de classes d'idéaux de k, et où R est le régulateur de k, est égal asymptotiquement à $\log |d|^{\frac{1}{2}}$, où d est le discriminant de k, quand $|d| \to \infty$. On sait que le résidu $r(k)$ de $\zeta(s, k)$ pour $s=1$ est $chR/|d|^{\frac{1}{2}}$, où c est compris entre des bornes positives ne dépendant que de n. Le résultat indiqué résulte de $\log r(k)/\log |d| \to 0$. L'auteur le démontre d'abord, par absurde, dans le cas galoisien. En vertu d'un lemme de Landau, on a

$$\limsup [\log r(k)/\log |d|] \leqq 0.$$

L'auteur tire de l'hypothèse $\liminf [\log r(k)/\log |d|] < -\epsilon$ ($\epsilon > 0$), grace à un lemme de Siegel limitant inférieurement $r(k)$ en fonction des n, $|d|$ et de quelque valeur réelle s, $0 < s < 1$ telle que $\zeta(s, k) \leqq 0$, l'existence d'un corps galoisien k de degré n (dont le discriminant soit d) tel que $\zeta(s, k)$ ait un zéro réel σ tel que $1-2\epsilon/n < \sigma < 1$. Il forme, ensuite, les composés $\Omega = kK$ de k avec tous les corps galoisiens K de degré n. Soient D le discriminant de K et $r(K)$, $r(\Omega)$ les résidus des $\zeta(s, K)$, $\zeta(s, \Omega)$ pour $s=1$. En vertu du résultat précédent de l'auteur, $\zeta(s, \Omega)/\zeta(s, k)$ est une fonction entière, d'où $\zeta(\sigma, \Omega) = 0$, d'où résulte, en vertu du lemme de Siegel et du théorème sur le discriminant des corps composés, que $\liminf [\log r(\Omega)/\log |D|] \geqq -\epsilon$. Or, $\zeta(s, \Omega)/\zeta(s, K)$ est une fonction entière $M(s)$ et $M(s)^n$ est, en vertu du théorème précédent de l'auteur, appliqué à Ω/K, un produit d'un nombre, borné en fonction de n, de facteurs $L(s; \chi, U/V)$, où les U/V sont des extensions abéliennes telles que $K \subseteq V \subset U \subseteq \Omega$, et où χ est un caractère du groupe de Galois de U/V autre que son 1-caractère. Toutes ces séries $L(s; \chi, U/V)$ sont des fonctions entières de s, et, en vertu d'un lemme de Landau,

$$|L(1; \chi, U/V)| \leqq \lambda_{n_V} \log^{n_V} |D_V \cdot N\mathfrak{f}_{U/V}|,$$

où n_V, D_V sont le degré et le discriminant de V, où $N\mathfrak{f}_{U/V}$ est la norme absolue du conducteur $\mathfrak{f}_{U/V}$ de U/V, et où λ_q ne dépend que de q. On a $\log r(K) = \log r(\Omega) - \log |M(1)|$, et, puisque D_V et, aussi, en vertu de la Führer-Diskriminantenformel de Hasse, $N\mathfrak{f}_{U/V}$ divisent $d^n D^n$, on a $\log |M(1)| \leqq c' \log \log |D|$, où c' ne dépend que de n, d'où $\limsup [\log |M(1)|/\log |D|] \leqq 0$ quand $|D| \to \infty$, et $\liminf [\log r(K)/\log |D|] \geqq -\epsilon$, contre l'hypothèse. Ainsi, $\liminf [\log r(k)/\log |d|] = 0$, d'où $\log r(k)/\log |d| \to 0$. Si k est un corps quelconque de degré n, et si d est le discriminant de k, Ω étant le corps de Galois de k, $M^*(s) = \zeta(s, \Omega)/\zeta(s, k)$ est une fonction entière, et l'auteur démontre d'une manière analogue que

$$\limsup [(\log r(\Omega) - \log r(k))/\log |d|] = \limsup \log M^*(1)/\log |d| \leqq 0,$$

d'où $\log r(k)/\log |d| \to 0$. *M. Krasner* (Paris).

Châtelet, Albert. Arithmétique des corps abéliens du troisième degré. Ann. Sci. École Norm. Sup. (3) **63** (1946), 109–160 (1947).

This is an elementary and complete discussion of cyclic cubic extensions of the rational field (abbreviated c.c.r.). Let R be the rational field, $R(\theta)$ a c.c.r., and form the two Lagrange resolvents $\rho = \theta + jS(\theta) + j'S^2(\theta)$ and $\rho' = \theta + j'S(\theta) + jS^2(\theta)$, where j and j' are the cube roots of unity and S an automorphism. Then there is a pair of conjugate numbers α, α' of $R(j)$ such that $\rho^3 = \alpha\rho'$, $\rho'^3 = \alpha'\rho$, $\rho^3 = \alpha^2\alpha'$, and θ satisfies

$$(*) \qquad (x-t)^3 - \tfrac{1}{3}\alpha\alpha'(x-t) - \tfrac{1}{27}\alpha\alpha'(\alpha+\alpha') = 0,$$

where $3t$ is the trace of θ. Conversely, for any rational t and any conjugate pair α, α' in $R(j)$, the equation (*) is irreducible and cyclic over R whenever $\alpha^2\alpha'$ is not a perfect cube in $R(j)$. This gives a parametrization of all cyclic cubic equations over R, with a simple condition for irreducibility, and throws description of the c.c.r. back on arithmetic in $R(j)$. There is a one-to-one correspondence between the set of all c.c.r. and a certain simply defined set of canonical pairs α, α'.

For each c.c.r., the author finds the value of the discriminant and an explicit basis for the integral domain and derives by elementary methods most of those results which follow easily from class field theory in this special case: number of c.c.r. with given discriminant, proof that they are all subfields of cyclotomic fields, and condition that a rational prime factors in a given c.c.r. Only two cases arise: discriminant divisible by 81 or prime to 3. Nothing is said about units. The author gives a parallel discussion of quadratic extensions of R and indicates how his methods might be extended to cyclic fields of higher degree.

G. Whaples (Bloomington, Ind.).

ANALYSIS

Fan, Ky. On positive definite sequences. Ann. of Math. (2) **47**, 593–607 (1946).

A sequence of complex numbers $\{a_n\}$ ($n=0, \pm1, \pm2, \cdots$) is called positive definite (p.d.) if $a_{-n} = \bar{a}_n$ and if the Hermitian form $\sum_{k,l=1}^{n} a_{k-l}\rho_k\bar{\rho}_l \geqq 0$, $n=1, 2, \cdots$. A sequence $\{X_n\}$ ($n=1, 2, \cdots$) of elements in Hilbert space is said to be stationary if the inner product (X_k, X_l) depends only on the difference $k-l$. The basis of the paper is the fact that, if $\{X_n\}$ is stationary, then $\{a_n\}$ defined by $a_{k-l} = (X_k, X_l)$ is p.d. and conversely any p.d. sequence $\{a_n\}$ may be represented in this way in terms of some appropriate stationary sequence. A systematic discussion of stationary sequences, including their property of the strong convergence of the mean $n^{-1}(X_1 + \cdots + X_n)$ as $n \to \infty$, affords a lucid derivation of the main properties of p.d. sequences leading up to the determination of their point spectrum, i.e., the determination of their almost periodic component.

I. J. Schoenberg (Philadelphia, Pa.).

Boas, Ralph P. Sur les suites vérifiant des inégalités portant sur leurs différences. C. R. Acad. Sci. Paris **224**, 1683–1685 (1947).

This note contains a generalization of a theorem due to Y. Tagamlitzki [same C. R. **223**, 940–942 (1946); these Rev. **8**, 259]. Let $0<\cdots<q_{-1}<q_0<q_1<\cdots<1$, $c_\nu\geqq 0$, $\sum c_\nu<\infty$. If the complex sequence $\{a_k\}$ satisfies the inequalities

$$|\Delta^n a_k|\leqq\sum_{\nu=-\infty}^{\infty}c_\nu\Delta^n q_\nu^k=\sum_{\nu=-\infty}^{\infty}c_\nu q_\nu^k(1-q_\nu)^n;\qquad k,n=0,1,2,\cdots,$$

then $a_k=\sum_{-\infty}^{\infty}d_\nu q_\nu^k$ with $|d_\nu|\leqq c_\nu$. The proof is based on the Hausdorff moment problem. In the case considered by Tagamlitzki, only one of the q_ν's is different from zero. As a corollary it is noted that if $f(x)$ is defined for $x\geqq 0$ and has derivatives of all orders and if $|f^{(n)}(x)|\leqq\sum_{\nu=0}^{\infty}c_\nu r_\nu^n e^{-r_\nu x}$, $0\leqq x<\infty$, where $r_\nu\to\infty$, $c_\nu\geqq 0$, and $\sum c_\nu<\infty$, then $f(x)=\sum_{\nu=0}^{\infty}d_\nu e^{-r_\nu x}$, $|d_\nu|\leqq c_\nu$. Finally the author gives a result which has a bearing on the trigonometric moment problem. If $\{a_k\}$ is a complex sequence, if for each N and each complex sequence $\{x_k\}$ one has

$$\left|\sum_{-N}^{N}\sum_{-N}^{N}a_{n-m}x_n\bar{x}_m\right|\leqq\left|\sum_{-N}^{N}x_n e^{inq}\right|^2,\qquad 0<q<2\pi,$$

then $a_n=Ae^{inq}$, $|A|\leqq 1$, $n=0,\pm1,\pm2,\cdots$. *E. Hille.*

Klingst, Anna. Eine Verallgemeinerung der Euler-Maclaurin'schen Reihe und der Bernoulli'schen Zahlen. Akad. Wiss. Wien, S.-B. IIa. **150**, 221–256 (1941).

The starting point is a quadrature formula of the type

$$(1)\qquad\int_{a_0}^{a_n}f(x)dx=\sum_{0}^{n}L_j f(a_j)+R,$$

where $a_0<\cdots<a_n$, $a_j+a_{n-j}=a_0+a_n$, $L_{n-j}=L_j$, and such that (1) is exact for polynomials of degree not exceeding m. If we assume for simplicity that m is odd, then the remainder R may be expressed in the form

$$(2)\qquad R=\int_{a_0}^{a_n}\phi(x)f^{(m+1)}(x)dx=-\int_{a_0}^{a_n}\phi'(x)f^{(m)}(x)dx,$$

where $\phi(x)$ is an even function about the midpoint $\frac{1}{2}(a_0+a_n)$, vanishing outside (a_0,a_n), composed of polynomial arcs of degree $m+1$ which join at the points a_i in such a way as to result in a function having $m-1$ continuous derivatives for all x. Since $\phi'(x)$ is odd about the midpoint, a certain properly adjusted primitive will be even and have its average zero in the range (a_0,a_n). In turn, its primitive will be odd and again a properly adjusted primitive will be even and of average value zero, and so on. Integrating by parts successively the second integral (2) by means of the sequence of primitives of ϕ' just mentioned, the author obtains the following generalization of the Euler-Maclaurin series:

$$(3)\qquad\int_{a_0}^{a_n}f(x)dx=\sum_{0}^{n}L_j f(a_j)-C_1\left(f^{(m)}(a_n)-f^{(m)}(a_0)\right)-C_3\left(f^{(m+2)}(a_n)-f^{(m+2)}(a_0)\right)-\cdots,$$

whose coefficients in turn define the higher Bernoulli numbers. The main point of the paper is the question as to when the odd function $\phi'(x)$, of (2), vanishes only at the midpoint of the range $a_0<x<a_n$ (the so-called condition S). For then it is easy to see that the remainder of (3), if this series is broken off at any point, may be estimated in the usual manner by the first mean-value theorem. In particular it is shown that the condition S is satisfied by Gauss's quadrature formulae and by the Cotes formulae of $n+1$ base points for $n=1,2,\cdots,10$. [The subject of the present paper has already been investigated by J. V. Uspensky in two papers [Bull. Amer. Math. Soc. **40**, 871–876 (1934); Trans. Amer. Math. Soc. **37**, 381–396 (1935)]. The first paper establishes the author's result concerning Gauss's formulae, while in the second paper Uspensky shows that all of Cotes's formulae satisfy the condition S.] *I. J. Schoenberg.*

Hadwiger, H. Der Begriff der Ultrafunktion. Vierteljschr. Naturforsch. Ges. Zürich **92**, 31–42 (1947).

The problem of differentiation and integration of fractional order involves defining a linear single-valued operator J_α on a suitable function space such that $J_\alpha J_\beta=J_{\alpha+\beta}$, $J_0[f]=f$, $J_{-1}[f]=f'$, $-\infty<\alpha<\infty$. The author overcomes the inherent difficulties of the problem by an enlargement of the function space in which the basic operations are to be defined. An ordinary analytic function is defined by the values of its derivatives at a point, for instance, at the origin; similarly an ultrafunction

$$\mathfrak{F}(z)=\{\cdots,a_{-2},a_{-1},a_0,a_1,a_2,\cdots\}$$

is defined by the values of the derivatives a_n, $n\geqq 0$, and of the integrals a_n, $n<0$, at the origin. Every ultrafunction can be written as the sum of two ordinary power series $\mathfrak{F}(z)=f(z)+jf^*(z)$, $f(z)=\sum_0^\infty a_n z^n/n!$, $f^*(z)=\sum_1^\infty a_{-n}z^n/n!$, where the commutator j is defined by

$$j\mathfrak{F}(z)=\{\cdots,a_2,a_1,a_0,a_{-1},a_{-2},\cdots\}$$

and $f(z)$, $f^*(z)$ are called the real and the ultra components of $\mathfrak{F}(z)$. Three subsets of the class of ultrafunctions are singled out for special consideration: the polynomial class P for which the components are polynomials, the real class R of functions with vanishing ultra components, and the Hilbert class H for which $\|\mathfrak{F}\|^2=\sum_{-\infty}^{\infty}|a_n|^2<\infty$. Various operations are defined for ultrafunctions and have a meaning for at least one of the classes H, P or R. Addition and scalar multiplication are defined in the obvious manner. The product of $\{a_n\}$ and $\{b_n\}$ is $\{c_n\}$ with $c_n=\sum_{\nu=0}^{\infty}\binom{n}{\nu}a_\nu b_{n-\nu}$ if the series are convergent for $n<0$. This product exists and coincides with the ordinary product on R. Similarly convolution is defined by $\{a_n\}*\{b_n\}=\{c_n\}$ with $c_n=\sum_{\nu=-\infty}^{\infty}a_\nu b_{n-\nu-1}$, if the series are convergent; it exists for functions in H or R. The author also defines a Laplace transform and proves convolution and product rules. He introduces the operator J_α in two steps. For integral values of α he sets $J_m\{a_n\}=\{a_{n-m}\}$ and proves rules for operating on convolutions and products. The final step is the introduction of the one-parameter family of ultrafunctions in H,

$$\mathfrak{T}_\alpha(z)=\left\{(-1)^n\frac{\sin\alpha\pi}{\pi}\frac{1}{n+\alpha-1}\right\}$$

with $\|\mathfrak{T}_\alpha\|=1$ and $\mathfrak{T}_\alpha(z)*\mathfrak{T}_\beta(z)=\mathfrak{T}_{\alpha+\beta}(z)$, in terms of which the operator J_α is defined by

$$J_\alpha(\mathfrak{F})=\mathfrak{T}_\alpha(z)*\mathfrak{F}(z)=\left\{(-1)^n\frac{\sin\alpha\pi}{\pi}\sum_{k=-\infty}^{\infty}(-1)^k\frac{a_k}{k+\alpha-n}\right\}$$

for $\mathfrak{F}\in H$. Then $J_\alpha(\mathfrak{F})\to J_m(\mathfrak{F})$ when $\alpha\to m$. The operator J_α defines a unitary transformation on H as has been observed by M. Riesz [Math. Z. **27**, 218–244 (1927)] and E. C. Titchmarsh [Proc. London Math. Soc. (2) **22**, no. 5, iii (1924)]. The operator J_α has all the required properties and gives a solution of the author's problem for the class H. [Reviewer's remark. The formula defining J_α is valid also for complex values of the parameter; so is the group property, and the operator J_α is an entire function of α.]

E. Hille (New Haven, Conn.).

Theory of Sets, Theory of Functions of Real Variables

Sierpinski, Waclaw. Sur les types d'ordre de puissance du continu. Revista Ci., Lima **48**, 305–307 (1946).

An effective construction is given for a class of $2^{\mathfrak{c}}$ sets, each containing $\mathfrak{c}$ real numbers, no two of which are ordinally similar when taken in natural order. It follows that the inequality $\overline{\overline{T}} \geqq 2^{\mathfrak{c}}$, where T is the class of all different order-types with cardinal $\mathfrak{c}$, can be established without an appeal to the axiom of choice; the opposite inequality has not been established without appealing to this axiom.

J. Todd (London).

Sierpiński, Wacław. Les correspondances multivoques et l'axiome du choix. Fund. Math. **34**, 39–44 (1947).

D. König [Fund. Math. **8**, 114–134 (1926)], using the axiom of choice, showed that if an (n, n) correspondence exists between two sets then a (1, 1) correspondence can be set up between these sets in such a way that the correspondent in it of an element is one of its n correspondents in the (n, n) correspondence. It is now shown that a proof of this result in the case $n=2$ which does not appeal to the axiom of choice would imply a proof, without use of this axiom, of the existence of a set not measurable-L. It is shown that the special case $n=2$ of König's result is a consequence of the special case of the axiom of choice concerned with selection from classes of couples. *J. Todd.*

Sierpiński, Wacław. Sur un espace complet qui n'admet pas le théorème de Souslin. Fund. Math. **34**, 66–68 (1947).

An analytic set in a metric space M can be defined (for instance) as the result of an analytic operation on sets closed in M or as the continuous image of a Borel set in M. These definitions are not in general equivalent in the case of a complete space which is only locally separable. A complete locally separable space is constructed in which the following theorem of Souslin is false, no matter which of these two definitions is used: a set which is analytic and whose complement is analytic is a Borel set. *J. Todd.*

Sierpiński, Wacław. Sur un problème concernant le crible de M. Lusin. Fund. Math. **34**, 69–71 (1947).

A sieve can be regarded as a function $E(r)$ associating a set $E(r)$ with each rational r. Its nucleus is the set $K_r(E(r)) = \sum\prod_i E(r_i)$, where the summation is over all decreasing sequences $\{r_i\}$. Consider $E(u, v)$, a function associating a set $E(u, v)$ with each pair of rationals u, v. Write $E(u) = K_v(E(u, v))$, $E = K_u(E(u))$. Kuratowski suggested the problem: do there exist functions $\phi(r)$, $\psi(r)$, associating rationals with the rational r, such that if $E(r) = E(\phi(r), \psi(r))$ then $E = K_r(E(r))$? It is shown that the answer is in the negative.

Denote by $K(\Phi)$ the class of all sets $K_r(E(r))$ where the $E(r)$ are sets of a class Φ. It is known that $KK(\Phi) = K(\Phi)$ whenever Φ is a (finitely) multiplicative class. This result is not, however, true for arbitrary classes. *J. Todd.*

Sierpiński, Wacław. Un théorème sur les puissances des ensembles. Fund. Math. **34**, 72–74 (1947).

The following theorem is established without appeal to the axiom of choice: if M is a class with cardinal $\mathfrak{m}$ and N a subclass with cardinal $\mathfrak{n}$ and if $\mathfrak{p}$ satisfies $\mathfrak{m} \geqq \mathfrak{p} \geqq \mathfrak{n}$ then there exists a class P with cardinal $\mathfrak{p}$ and such that $M \supset P \supset N$. *J. Todd* (London).

Sierpiński, W. Sur un problème de triades. C. R. Soc. Sci. Varsovie **33–38**, 13–16 (1946). (French. Polish summary)

A set E is said to have the property of Steiner if there exists a family $\mathfrak{F}$ of three-element subsets of E such that any two-element subset of E lies in exactly one set of the family $\mathfrak{F}$. It is known that a finite set has the property of Steiner if and only if its cardinality is of the form $6k+1$ or $6k+3$. [See E. Netto, Lehrbuch der Combinatorik, 2d ed., Teubner, Leipzig, 1927, pp. 202–218.] The author proves by using the axiom of choice that every infinite set has the property of Steiner. *W. Gustin* (Bloomington, Ind.).

Choquet, Gustave. Application à la théorie des réseaux, d'un théorème sur la structure des permutations d'un ensemble. J. Math. Pures Appl. (9) **25**, 161–172 (1946).

Soit E un ensemble abstrait quelconque, fini ou infini, contenant au moins deux éléments. On dit qu'un sous-ensemble quelconque $\mathfrak{A}$ de l'ensemble $\mathcal{S}$ des couples non ordonnés d'éléments distincts de E, est un "réseau" défini sur E ($\mathfrak{A}$ peut être vide ou identique à $\mathcal{S}$). Soit P une permutation quelconque de E, $\mathcal{P}$ la permutation de $\mathcal{S}$ associée à P. Deux réseaux $\mathfrak{A}_1$, $\mathfrak{A}_2$ définis sur E sont "isomorphes," s'il existe une permutation P de E telle que $\mathfrak{A}_2 = \mathcal{P}(\mathfrak{A}_1)$. Un réseau $\mathfrak{A}$ défini sur E est "réciproque," lorsque $\mathfrak{A}$ est isomorphe à son "complémentaire" $\mathfrak{B} = \mathcal{S} - \mathfrak{A}$. L'auteur montre que, pour qu'on puisse définir un réseau réciproque sur E, il faut et il suffit que E soit infini, ou composé de $4n$ ou $4n+1$ éléments (n entier, $n \geqq 1$). Il montre comment on peut définir tous les réseaux réciproques sur E. Enfin il fait une brève étude de la structure de ces réseaux.

T. Viola (Rome).

Cesari, L. Un criterio per la misurabilità degli insiemi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 1256–1263 (1946).

Let $f(x, y)$ be a (real-valued) function defined in the square Q: $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. According to an example constructed by Sierpiński, $f(x, y)$ may be a measurable function of y for every x in the interval $0 \leqq x \leqq 1$ without being measurable in Q with respect to both of the variables x, y. The purpose of the present note is to develop an additional sufficient (but not necessary) condition to enforce the measurability of $f(x, y)$ in Q. The following concepts are used. A plurinterval, in $0 \leqq x \leqq 1$, is a set represented as a (finite or infinite) sum of open intervals without common points. If x_0 is a point and E is a set in the interval $0 \leqq x \leqq 1$, then x_0 is termed a bilateral cluster-point of E provided that for every $\epsilon > 0$ there exist points x_1, x_2 in E such that $x_0 - \epsilon < x_1 < x_0 < x_2 < x_0 + \epsilon$. The main result of the paper may then be stated as follows. Suppose $f(x, y)$ satisfies in Q the following two conditions. (P_1) For almost every $\bar{x}$, $0 \leqq \bar{x} \leqq 1$, $f(\bar{x}, y)$ is a measurable function of y for $0 \leqq y \leqq 1$. (P_2) For every positive integer n there exists a plurinterval Δ_n in $0 \leqq x \leqq 1$ of total length less than $1/n$ such that, if N_n is the set of those points of the set $E_n = (0, 1) - \Delta_n$ which are bilateral cluster-points of E_n, then for every $\bar{x} \in N_n$ and for every $\bar{y}$, $0 \leqq \bar{y} \leqq 1$, at least one of the limits $f_r(\bar{x}, \bar{y}) = \lim f(x, \bar{y})$, $x \to \bar{x} + 0$, $x \in E_n$, and $f_l(\bar{x}, \bar{y}) = \lim f(x, \bar{y})$, $x \to \bar{x} - 0$, $x \in E_n$, exists and $f_r(\bar{x}, \bar{y}) = f(\bar{x}, \bar{y})$ or $f_l(\bar{x}, \bar{y}) = f(\bar{x}, \bar{y})$, respectively. Theorem: if (P_1) and (P_2) hold, then $f(x, y)$ is measurable in Q. *T. Radó* (Columbus, Ohio).

Tibaldo, L. Un teorema sulle funzioni misurabili rispetto ad una e continua rispetto ad un'altra variabile. Applicazioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 2, 146–152 (1947).

In connection with the theory of ordinary differential equations, there arises the question: if $f(x, y)$ is defined in a rectangle R: $a \leqq x \leqq b$, $c \leqq y \leqq d$, and $f(x, y)$ is measurable for every y as a function of x and continuous for every x as a function of y, what can be said about continuity properties of $f(x, y)$ with respect to both variables? The main result of the note is as follows. Given $\epsilon > 0$, there exists a measurable subset E in $a \leqq x \leqq b$ of measure greater than $b-a-\epsilon$, such that $f(x, y)$ is uniformly continuous with respect to y on the set R^* where $x \in E$, $c \leqq y \leqq d$.

T. Radó (Columbus, Ohio).

Szymański, Piotr. Sur une correspondance effective entre l'ensemble des fonctions continues et un ensemble de nombres réels. Bull. Sci. École Polytech. Timişoara 11, 185–195 (1944).

The author constructs an effective one-to-one correspondence between the set of all continuous functions (which are real-valued, the domain being any separable topological space S) and a certain set H of real numbers. Then he proves some properties of this correspondence (which certainly is discontinuous) and of the set H. For instance, H is totally disconnected and, if S is a Euclidean space, also dense in $(0, 1)$. *A. Rosenthal* (Lafayette, Ind.).

Tambs Lyche, R. L'identité de deux fonctions continues sans dérivée. Norske Vid. Selsk. Forh., Trondhjem 16 (1943), no. 29, 106–107 (1944).

In an earlier paper [same Forh. 12, 45–48 (1939); these Rev. 1, 109] the author showed that van der Waerden's continuous nondifferentiable function

$$(1) \qquad \sum_{n=0}^{\infty} 2^{-n} g(2^n x)$$

($g(x)$ is the distance of x from the nearest integer) can be represented in the form

$$(2) \qquad \sum_{n=1}^{\infty} \frac{c_n - 2(n-1)}{2^{c_n}},$$

where c_n is the increasing sequence of natural numbers defined by $x = x_0 + \sum_{n=1}^{\infty} 2^{-c_n}$ (x_0 integral). A new simple proof of this result is given. *F. A. Behrend* (Melbourne).

Tambs Lyche, R. Sur une série trigonométrique. Norske Vid. Selsk. Forh., Trondhjem 16 (1943), no. 30, 108–110 (1944).

The author shows that the series

$$\tfrac{1}{2} - 2\pi^{-2} \sum_{n=1}^{\infty} (n\nu)^{-1} \cos 2\pi n x$$

(where ν denotes the greatest odd divisor of n) is equal to the series (2) of the preceding review and hence represents van der Waerden's function (1). This result, however, follows more directly by substituting the Fourier series of $g(x)$,

$$g(x) = \tfrac{1}{4} - 2\pi^{-2} \sum_{m=0}^{\infty} (2m+1)^{-2} \cos (2m+1) 2\pi x,$$

into (1). *F. A. Behrend* (Melbourne).

Cioranescu, Nicolas. La derivée moyenne d'une fonction et certaines équations fonctionnelles. Disquisit. Math. Phys. 1, 29–34 (1940).

The author considers, as a specialisation of more general formulae, the expression $D_\alpha f(x) = 6\alpha^{-3} \int_0^\alpha (2t-\alpha) f(x+t)\,dt$, expressible as $f'(x+\theta\alpha)$ $(0<\theta<1)$ if $f(x)$ is differentiable. He gives some transformations of the defining formula and a generalisation corresponding to the nth derivative. He states that the only summable functions satisfying $D_\alpha f(x) = 0$ are those constant almost everywhere, but does not make it clear whether D_α is to vanish for one fixed α or for all $\alpha \neq 0$. The reviewer finds the theorem false in the former interpretation and true in the latter. *A. J. Ward.*

Myshkis, A. On the existence of the complete differential on the boundary of a plane domain. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 359–392 (1946). (Russian. English summary)

An expansion of a paper by the same author [C. R. (Doklady) Acad. Sci. URSS (N.S.) 48, 82–85 (1945); these Rev. 7, 378]. *A. J. Ward* (Cambridge, England).

Marchaud, A. Sur les propriétés différentielles du premier ordre des surfaces simples de Jordan et quelques applications. Ann. Sci. École Norm. Sup. (3) 63 (1946), 81–108 (1947).

Fundamental theorems are obtained concerning the tangents (limits of secants M_1M_1' as M_1 and M_1' tend to M_0) and half-tangents (limits of rays M_0M_1 as M_1 tends to M_0) to a simple Jordan surface S in 3 dimensions. The tangents at M_0 form the tangent bundle (faisceau des tangentes = paratingent of Bouligand) $\mathfrak{I}(S; M_0)$ and the half-tangents form the derivate bundle (faisceau dérivé = contingent) $\mathfrak{D}(S; M_0)$. It is assumed that M_0 is an interior point of S at which there is a line not in $\mathfrak{I}(S; M_0)$, so that S may be represented near M_0 as $z = f(x, y)$, where f satisfies a Lipschitz condition. Among the results are the following. (1) The lines not in $\mathfrak{I}(S; M_0)$ form two symmetric cones $\theta(M_0), \theta'(M_0)$ which are convex. (2) The rays not in $\mathfrak{D}(S; M_0)$ form two cones not necessarily convex but having a kind of convexity relation to $\theta(M_0)$ and $\theta'(M_0)$. (3) If $D(M_0, \alpha)$ is a derivate at M_0 of a section of S by a half-plane parallel to the z-axis and making an angle α with the x-axis then the set of all $D(M_0, \alpha)$ for $-\pi \leqq \alpha < \pi$ is $\mathfrak{D}(S; M_0)$. (4) If, for a fixed direction α, any $D(M, \alpha)$ is discontinuous at M_0 as $M \to M_0$, then the derivates in all directions (except possibly two opposed directions) are all discontinuous at M_0. If, on the other hand, the derivate bundle is continuous at M_0 then the tangent bundle there is a plane.

Finally, these ideas are applied to show that, if S is a portion of the surface $z = f(x, y)$ which projects into a convex domain on $z = 0$, and if all planes parallel to the z-axis cut S in convex curves (or, equivalently, if S is a portion of a surface of the second order) then either (a) S is a part of the frontier of a convex domain (if all sections are convex in the same sense) or (b) S is a part of a nondegenerate ruled quadric. *U. S. Haslam-Jones* (Oxford).

Morse, Anthony P. Perfect blankets. Trans. Amer. Math. Soc. 61, 418–442 (1947).

The definition of a "perfect blanket" may be summarised as follows: A is a set in a metric space S; $F(x)$, defined for $x \in A$, is a family of bounded subsets of S such that, for

fixed x, the lower bound for $\beta \varepsilon F(x)$ of the diameter of $\beta + (x)$ is 0; then F is said to be a blanket. Let ϕ be a nonnegative function of sets, finite for bounded sets, such that $\phi(\sum\beta) \leqq \sum\phi(\beta)$ for any enumerable infinity of sets β and $\phi(\beta_1+\beta_2) = \phi(\beta_1)+\phi(\beta_2)$ for any two separated sets β_1, β_2. Then F is a ϕ heavy blanket if all the sets in each $F(x)$ are Borelian and it is possible to select from $\sum F(x)$ an enumerable infinity of disjoint sets e_i so that $\phi\{A - \sum e_i\} = 0$; F is a ϕ strong blanket if F and all its sub-blankets are ϕ heavy. Finally, F is a perfect blanket if it is ϕ strong for all ϕ of the class defined above.

Translated into this language a theorem of A. Besicovitch [Proc. Cambridge Philos. Soc. **41**, 103–110 (1945); **42**, 1–10 (1946); these Rev. **7**, 10, 281] states that, in n-dimensional Cartesian space R^n, every blanket in which $F(x)$ contains all spheres with center x is perfect. In §§ 2–4 the author investigates properties of blankets and covering relations in a general metric space. In § 5 the space is R^n and it is shown that Besicovitch's theorem is true for more general blankets. A class of "star blankets" is defined which includes, for example, blankets of convex sets with restricted ratio of diameter to internal radius, and it is shown that every star blanket is a perfect blanket. *U. S. Haslam-Jones.*

de Possel, René. Sur la généralisation de la notion de système dérivant. C. R. Acad. Sci. Paris **224**, 1137–1139 (1947).

Let m be a given measure over a Borelian system $\mathfrak{Z}$, $\vec{\vartheta}$ any enumerably additive set-function over $\mathfrak{Z}$, of bounded variation into a complete normed vector-space. We associate to each point x a filter $\mathfrak{F}_x$ of nonnegative $\mathfrak{Z}$-measurable real functions. Such a filter-system is called weakly deriving if, for every Lipschitz function $\vec{\vartheta}$,

$$\vec{D} = \lim_{\mathfrak{F}_x} \int f d\vec{\vartheta} \Big/ \int f dm$$

exists almost everywhere (m), and yields $\vec{\vartheta}$ by m-integration; strongly deriving if the same holds for every m-absolutely continuous $\vec{\vartheta}$. Conditions are stated (without proof) for a given filter-system to have either property.

A. J. Ward (Cambridge, England).

de Possel, René. Sur les systèmes dérivants et l'extension du théorème de Lebesgue relatif à la dérivation d'une fonction à variation bornée. C. R. Acad. Sci. Paris **224**, 1197–1198 (1947).

With the notation of the paper reviewed above, a filter-system is called completely deriving if, for every $\vec{\vartheta}$ of bounded variation, enumerably additive over $\mathfrak{Z}$, $\vec{D}$ exists almost everywhere (m), and $\int \vec{D} dm - \vec{\vartheta}$ is m-singular. A condition is stated for this to hold, and it is remarked (i) that this includes the classical theorem for linear Lebesgue measure, (ii) that a filter-system may be strongly while not completely deriving. *A. J. Ward* (Cambridge, England).

Rogosinski, W. W. Measure and integral. Edinburgh Math. Notes no. **36**, 1–12 (1947).

It is the purpose of this expository note to sketch the build up of both the Riemann and the Lebesgue integral on the common geometrical basis of "measure" and thus to make evident to the uninitiated reader the striking advantages of the new integral. *Extract from the paper.*

Dubrovsky, V. Remarks to my paper "On some properties of completely additive set functions and passing to the limit under the integral sign." Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **11**, 101–104 (1947). (Russian. English summary)

The results of the author's earlier note [same Bull. **9**, 311–320 (1945); these Rev. **7**, 280] on sequences of completely additive measures defined over a Borel field of sets are extended to the case where the Borel field need not contain the individual points of the space. For functions defined on the Borel sets of a Euclidean space, examples are given showing the relations between various types of convergence of sequences of set-functions. *M. M. Day.*

Henstock, R. On interval functions and their integrals. J. London Math. Soc. **21**, 204–209 (1946).

L'auteur rappelle d'abord la définition des intégrales supérieure et inférieure de Burkill d'une fonction d'intervalle $g(I)$ non nécessairement additive; il remarque que si ces intégrales sont égales et finies sur un intervalle I, leur valeur commune possède l'additivité finie dans I. Il propose ensuite de modifier la définition de manière à obtenir des intégrales supérieure et inférieure qui soient toujours additives et égales à $g(I)$ si cette fonction est additive. Pour cela il introduit pour une fonction $g(I)$ dont les intégrales de Burkill ordinaires sont finies, des points dits "singuliers" qui sont en infinité dénombrable. En n'utilisant que des suites de divisions dans lesquelles chacun de ces points intervient toujours à partir d'un certain rang, le résultat est obtenu. L'auteur remarque que S. Pollard [Quart. J. Math. **49**, 73–138 (1921)] avait proposé une convention analogue pour l'intégrale de Riemann-Stieltjes. *R. de Possel.*

Cotlar, Mischa, and Frenkel, Yanny. On nonadditive set functions and generalization of the decomposition of Vallée Poussin. Univ. Nac. Tucumán. Revista A. **5**, 227–253 (1946). (Spanish)

Generalizing ideas due to R. C. Young [Math. Z. **29**, 171–216, 217–233 (1928)] and to A. Kolmogoroff [Math. Ann. **103**, 654–696 (1930)], the authors introduce, for real functions of abstract sets, two semi-order relations which they term inequality (σ) and inequality (λ): these reduce to $\leqq$ when the set-functions are additive and when they are additive and singular, respectively. It is shown that there exist greatest additive functions φ_σ and φ_λ in these two relationships to a given set-function $f(e)$ of bounded variation; these additive functions are termed lower integrals (σ) and (λ). The corresponding lower integrals of $\min(f, 0)$ are denoted by $\underline{W}$ and $\underline{E}$ and termed lower variation and lower saltus of f. Extensions of the classical decomposition theorems of Lebesgue and de la Vallée Poussin are given: each of these splits up into an upper and a lower theorem. For instance, lower net-derivatives $\underline{D}f$ are defined with respect to the measure $\mu(e)$ and it is shown that, for an f of bounded variation, if E is the set in which $\underline{D}f = \pm\infty$, then $\mu(E) = 0$ and $\varphi_\sigma(e) = \varphi_\sigma(eE) + \int_e \underline{D}f\, d\mu$.

L. C. Young (Columbus, Ohio).

Helsel, R. G. A theorem on surface area. Trans. Amer. Math. Soc. **61**, 443–453 (1947).

Let $F(T)$ be a nonnegative lower semicontinuous functional whose argument T ranges over the continuous transformations from a unicoherent Peano space P to a metric space. T. Radó [Trans. Amer. Math. Soc. **58**, 420–454 (1945); these Rev. **7**, 282] established a necessary and sufficient condition for $F(T)$ to be "cyclically additive." By

applying this condition to the case in which P is either a 2-cell or a 2-sphere and $F(T)$ is the Lebesgue area or the Radó lower area of the surface with the representation T, the author proves at Radó's suggestion that in these cases $F(T)$ is invariant for parametric representations which are equivalent in the sense of Kerékjártó. (This sense is more general than that of Fréchet.) This invariance was previously proved in the case of the Lebesgue area $F(T)$ and of a 2-sphere P by J. W. T. Youngs [Ann. of Math. (2) **45**, 753–785 (1944); these Rev. **6**, 278]. *L. C. Young.*

Busemann, Herbert. Intrinsic area. Ann. of Math. (2) **48**, 234–267 (1947).

Let H_δ^n be the n-dimensional Hausdorff measure associated with the metric δ. It is not excluded that $\delta(x, y) = \infty$, or that $\delta(x, y) = 0$ with $x \neq y$. The author first considers the case in which δ is a Minkowskian metric metrizing n-space and proves that $H_\delta^n = cL_n$, where L_n is n-dimensional Lebesgue measure and c is the ratio of the L_n measures of the ordinary Euclidean and the δ Minkowskian unit spheres. Just as in the previously known Euclidean case, the proof hinges on the isodiametric inequality. The author shows that of all sets of a given δ diameter, the δ sphere has the largest L_n measure.

The next section concerns itself with "intrinsic area." Consider all triples (α, β, x) such that α and β are metrics, the space metrized by α is α arcwise connected and x is a function on the space metrized by α to the space metrized by β, continuous with respect to α and β. A new metric δ, metrizing the same space as α, is associated with the triple (α, β, x) as follows: $\delta(p, q)$ is the greatest lower bound of the β lengths of all curves of the type xc, where c is an α curve joining p and q. Then $A^n(\alpha, \beta, x) = H_\delta^n(\text{domain } x)$ is the n-dimensional "intrinsic area" of the triple (α, β, x). Of the many geometrical properties which the author proves in order to justify his use of the term "area," we shall mention here only the fact that $A^n(\alpha, \beta, x)$ agrees with the various classical definitions in case α and β are Euclidean, Riemannian or Finsler metrics and both x and its inverse satisfy Lipschitz conditions.

Since A^n has, like most areas, very delicate additivity properties, the author considers also another function B^n. Let (α, β, x) be a triple as above and let $\epsilon > 0$. The metric δ_ϵ, metrizing the same space as α, is defined as follows: $\delta_\epsilon(p, q)$ is the greatest lower bound of all numbers of the form $\sum_{i=1}^{k} \beta(x(r_{i-1}), x(r_i))$, where $r_0 = p$, $r_k = q$ and $\alpha(r_{i-1}, r_i) < \epsilon$ for $i = 1, \cdots, k$. Then $B^n(\alpha, \beta, x) = \lim_{\epsilon \to 0} H_{\delta_\epsilon}^n(\text{domain } x)$. It occurs to the reviewer that

$$B^n(\alpha, \beta, x) = \int_{\text{range } x} f dH_\beta^n,$$

where the multiplicity function f is defined as follows. A set S shall be called α chain connected if and only if given $p \in S$, $q \in S$, $\epsilon > 0$, there exist points $r_0, \cdots, r_k \in S$ such that $r_0 = p$, $r_k = q$, $\alpha(r_{i-1}, r_i) < \epsilon$ for $i = 1, \cdots, k$. For each $y \in$ range x, $f(y)$ is the number (possibly ∞) of all maximal α chain connected subsets of the counterimage of y under the map x.

H. Federer (Providence, R. I.).

Theory of Functions of Complex Variables

Germay, R.-H.-J. Sur une formule de Lagrange généralisée. Bull. Soc. Roy. Sci. Liége **14**, 392–398 (1945).

Consider the equation $(z-a)(z-b) - \alpha f(z) = 0$, where a and b lie within a circle C and $f(z)$ is analytic at all points within C and on its circumference. The author shows that for $|\alpha|$ sufficiently small (1) has two roots within C. His primary concern is the obtaining of power series in α for these roots and for certain analytic functions of them.

T. Fort (Athens, Ga.).

Germay, R.-H.-J. Sur la série de Lagrange-Bürmann et sa généralisation. Bull. Soc. Roy. Sci. Liége **14**, 464–470 (1945).

In this paper the results of the note reviewed above are extended to $(z-a)^p(z-b)^q - F(\alpha, \varphi(z)) = 0$. *T. Fort.*

Anghelutză, Th. Sur le développement d'une fonction holomorphe dans une série de polynomes. Bull. Sci. École Polytech. Timişoara **11**, 179–180 (1944).

Anghelutză, Th. Le développement d'une fonction analytique d'après les fonctions fondamentales d'un noyau. Bull. Sci. École Polytech. Timişoara **12**, 14–16 (1945).

Let the rectifiable simple closed curve C (for simplicity surrounding the origin) be interior to a given domain D. Let C_1 be the transform of C under a stretch of ratio $1/r$, center the origin, where the constant $r < 1$ is chosen small enough so that C_1 also lies interior to D; so C, C_1 have the respective equations $C: \rho = r\rho(\theta)$; $C_1: \rho = \rho(\theta)$. From Cauchy's integral formula one obtains, for x on C and for every function $f(z)$ analytic in D, with $z = \rho(\theta)e^{i\theta}$,

$$f(x) = \int_0^{2\pi} k(\varphi, \theta) f(z) d\theta; \quad k(\varphi, \theta) = \frac{e^{i\theta}}{2\pi i} \frac{\rho'(\theta) + i\rho(\theta)}{\rho(\theta)e^{i\theta} - r\rho(\varphi)e^{i\varphi}};$$

and from $k(\varphi, \theta)$ a new kernel $N_1(\varphi, \theta) = \int_0^{2\pi} k(\varphi, \omega)\bar{k}(\theta, \omega) d\omega$ is formed ($\bar{k}$ the conjugate imaginary of k). The eigenfunctions for N_1 define a sequence of functions $\{\varphi_n(z)\}$, analytic in and on C, and it is shown that $f(z)$ has the expansion $f(z) = \sum_1^\infty a_n \varphi_n(z)$, uniformly convergent in and on C. The coefficients $\{a_n\}$ are expressed by means of an integral formula. *I. M. Sheffer* (State College, Pa.).

Morgan, Edith L. An application of the theory of the complex variable to the mapping of some potential functions. J. Franklin Inst. **243**, 309–322 (1947).

Simple combinations of point charges. Illustrations.

A. J. Macintyre (Aberdeen).

Guelfer, S. On the class of regular functions which do not take on any pair of values w and $-w$. Rec. Math. [Mat. Sbornik] N.S. **19(61)**, 33–46 (1946). (Russian. English summary)

The author considers the following two classes of functions: the class (C) of functions $w = f(z) = \sum_{n=1}^\infty a_n z^n$ which are regular in $|z| < 1$ and satisfy the condition that $f(z_1) \cdot f(z_2) \neq 1$ for $|z_1| < 1$, $|z_2| < 1$; the class (D) of functions $w = \varphi(z) = 1 + \sum_{n=1}^\infty \alpha_n z^n$ which are regular in $|z| < 1$ and satisfy $\varphi(z_1) + \varphi(z_2) \neq 0$ for $|z_1| < 1$, $|z_2| < 1$. If f belongs to (C), then $(1+f(z))/(1-f(z))$ belongs to (D); conversely if φ belongs to (D), $(\varphi(z)-1)/(\varphi(z)+1)$ belongs to (C). The author finds inequalities for the moduli of these functions and their derivatives and in the last section applies the results to obtain known properties of schlicht functions.

D. C. Spencer (Stanford University, Calif.).

* **Golusin, G. Estimates for analytic functions with bounded mean of the modulus.** Trav. Inst. Math. Stekloff **18**, 87 pp. (1946). (Russian. English summary)

Due to the wealth of results in this paper, the reviewer is able to give only an incomplete summary of the contents.

The author studies the class, denoted by H_1, of functions $f(z)$ analytic in $|z|<1$ for which

$$(2\pi)^{-1}\int_0^{2\pi}|f(re^{i\theta})|\,d\theta\leqq 1, \qquad 0\leqq r<1.$$

The methods used depend basically on the following result of Carathéodory and Fejér [Rend. Circ. Mat. Palermo **32**, 218–239 (1911)]. For every set of $n+1$ complex constants $c_0, c_1, \cdots, c_n$, not all zero, there exists a unique rational fraction of the form

$$(*) \qquad R(\zeta)=\lambda\frac{\bar{\alpha}_n+\cdots+\bar{\alpha}_0\zeta^n}{\alpha_0+\cdots+\alpha_n\zeta^n}, \qquad \lambda>0,$$

regular in $|\zeta|<1$ and possessing in its expansion about $\zeta=0$ the constants $c_0, c_1, \cdots, c_n$ as its first $n+1$ coefficients. Among all functions $f(\zeta)=c_0+c_1\zeta+\cdots+c_n\zeta^n+\cdots$ regular in $|\zeta|<1$ and possessing the preassigned constants as the first $n+1$ coefficients, $R(\zeta)$, and only $R(\zeta)$, gives the least value for $M_f=\text{l.u.b.}_{|\zeta|<1}|f(\zeta)|$. This value is λ and is equal to the largest positive root of a certain algebraic equation of the $2n$th degree formed from the constants $c_0, c_1, \cdots, c_n$. The author denotes by H_1^0 those functions of class H_1 which do not vanish in $|z|<1$; by H_1^+ those functions of class H_1 which can be represented in $|z|<1$ by an expansion of the form $(\sum_{n=0}^{\infty}b_nz^n)^2$ with real nonnegative b_n.

Continuing his earlier study of functions of class H_1 [Rec. Math. [Mat. Sbornik] N.S. **18**(60), 213–226 (1946); these Rev. **8**, 22], the author considers the least upper bounds of various expressions involving the coefficients c_k formed for all functions $f(z)=\sum_{k=0}^{\infty}c_kz^k$ of class H_1. The expressions studied are (1) $|\sum_{k=0}^{\infty}\gamma_kc_k|$ for preassigned complex numbers γ_k with $\sum_{k=0}^{\infty}|\gamma_k|<\infty$; (2) $\sum_{k=0}^{\infty}\rho_k|c_k|^p$, $p\geqq 1$, for preassigned real $\rho_k\geqq 0$ with $\sum_{k=0}^{\infty}\rho_k<\infty$; (3) $|\sum_{k=0}^{\infty}\rho_kc_k{}^p|$ for the same ρ_k and that determination of $c_k{}^p$ which is positive for positive c_k; (4) $\int_a^b|\sum_{k=0}^{\infty}\gamma_k(t)c_k|^p\,d\alpha(t)$ for preassigned complex functions $\gamma_k(t)$ continuous in $a\leqq t\leqq b$ with $\sum_{k=0}^{\infty}|\gamma_k(t)|$ uniformly convergent in $a\leqq t\leqq b$ and $\alpha(t)$ bounded, real, nondecreasing in $a\leqq t\leqq b$; (5) $\int_a^b(\sum_{k=0}^{\infty}\rho_k(t)|c_k|)^p\,d\alpha(t)$ for $\alpha(t)$ and p as above and $\rho_k(t)\geqq 0$, continuous, with $\sum_{k=0}^{\infty}\rho_k(t)$ uniformly convergent in $a\leqq t\leqq b$; and other analogous expressions. It is shown that the least upper bound of (1) is attained for a function of class H_1^0; of (2) and (3) for a function of H_1^+ (and for the same ρ_k and p the two bounds are equal); of (4) and (5) for a function of H_1^0 and H_1^+, respectively (and for $\gamma_k(t)=\rho_k(t)$ and the same $\alpha(t)$ and p the two bounds are equal). The corresponding problem for finite sums $|\sum_{k=0}^{n}\chi_kc_k|$ with preassigned $\gamma_0, \gamma_1, \cdots, \gamma_m$, not all zero, was solved by the author in the paper previously referred to. He now adds that if the γ_k are all real, then the bound is attained for a polynomial of the form $f(z)=(d_0+d_1z+\cdots+d_mz^m)^2$, where $\sum_{k=0}^{m}|d_k|^2=1$ and that if, in addition, the γ_k are nonnegative, then the d_k are nonnegative.

This leads the author to a solution of the "coefficient problem" for functions of class H_1, at least in a geometric form. The problem is: setting $c_k=x_k+iy_k$, $k=0, 1, \cdots, m$, and considering $x_0, y_0, \cdots, x_m, y_m$ as the $2m+2$ rectangular coordinates in a Euclidean space R_{2m+2} of $2m+2$ dimensions, what is the subset B of R_{2m+2} which consists of all such points $(x_0, y_0, \cdots, x_m, y_m)$ that the corresponding numbers c_k, $k=0, \cdots, m$, can be the first $m+1$ coefficients in the expansion of some function of class H_1, about $z=0$? Denoting by B^* that subset of R_{2m+2} which corresponds in this sense to all polynomials $f(z)=\sum_{k=0}^{2m}c_kz^k=(\sum_{k=0}^{m}d_kz^k)^2$ with the sole condition that $\sum_{k=0}^{m}|d_k|^2=1$, it is shown that B is the smallest closed convex body containing B^*. Since the condition $\sum_{k=0}^{m}|d_k|^2=1$ can be easily expressed in terms of the c_k, this solves the problem.

Various estimates for functions of class H_1 are made. Thus, if $f(z)\in H_1$, then for $n=0, 1, 2, \cdots$

$$|f^{(n)}(z)|\leqq n!\lambda_r^{(n)}/(1-r^2)^{n+1}, \qquad |z|=r,$$

in $|z|<1$, where $\lambda_r^{(n)}$ is the number λ appearing in (*) for a function $R(\zeta)$ whose expansion about $\zeta=0$ begins with the $n+1$ terms of the polynomial $(1+r\zeta)^{n+1}-r^{n+1}\zeta^{n+1}$. The inequality is sharp, equality being attained at $\zeta=z=r>0$ for

$$f(\zeta)=\left[\alpha_0+\alpha_1\frac{\zeta-z}{1-z\zeta}+\cdots+\alpha_n\left(\frac{\zeta-z}{1-z\zeta}\right)^n\right]\frac{1-r^2}{(1-z\zeta)^2},$$

where $\sum_{k=0}^{n}|\alpha_k|^2=1$. For any n, $\lambda_r^{(n)}$ is nondecreasing in r for $0<r<1$ and $\lim_{r\to 1}\lambda_r^{(n)}=\lambda_1^{(n)}\leqq 2^{n+1}$. Further results about $\lambda_r^{(n)}$ are given. Sharp estimates are proved for $|f(z)|$ when $f(z)$ is in H_1 and $|f(0)|$ has a preassigned value. Among other estimates the following generalization of the classical Fejér-Riesz inequality is given: if $\mu\geqq 2$ is an integer, $\eta=e^{2\pi i/\mu}$, and $f(z)$ is in H_1, then

$$\sum_{\nu=1}^{\mu}\int_0^1|f(\eta^\nu r)|\,dr\leqq\pi/\sin(\pi/\mu).$$

For $\mu>2$ the sign of equality is attained only for the function $f(z)=c/(1-\epsilon z^\mu)^{2/\mu}$ for suitable $c>0$ and ϵ, $|\epsilon|=1$. For $\mu=2$ this reduces to the Fejér-Riesz inequality. Generalizing earlier results of Fejér [Math. Ann. **97**, 104–123 (1926)] and Egerváry [Math. Ann. **99**, 542–561 (1928)], the author proves that if $f(z)=\sum_{k=0}^{\infty}c_kz^k$ is in H_1, setting $m(r)=\sum_{k=0}^{\infty}|c_k|r^k$, $0\leqq r<1$, then $m^{(n)}(r)\leqq n!\lambda_r^{(n)}/(1-r^2)^{n+1}$, $0\leqq r<1$, where $\lambda_r^{(n)}$ has the same meaning as before and $m^{(n)}(r)$ denotes the nth derivative of $m(r)$. The case $n=0$ is due to Egerváry. Furthermore, the following sharp estimates hold:

$$\sum_{k=0}^{\infty}\frac{|c_k|}{k+1}\leqq\pi, \quad \sum_{k=0}^{\infty}\frac{|c_{k\mu}|}{k\mu+1}\leqq\frac{\pi}{\mu\sin(\pi/\mu)}, \quad \sum_{k=0}^{\infty}\frac{|c_k|^2}{k+1}\leqq 1.$$

The first inequality is due to Fejér. Extending a result of Robinson [Bull. Amer. Math. Soc. **46**, 849–851 (1940); these Rev. **2**, 79], inequalities are found for $\int_0^{2\pi}|f(re^{i\theta})|^2d\theta$ and $\int_0^{2\pi}|f'(re^{i\theta})|\,d\theta$. One example should suffice for another type of result. For functions $f(z)=\sum_{n=0}^{\infty}c_nz^n\in H_1$ and for any sequence of positive integers n_ν, $\nu=0, 1, \cdots$, where $n_{\nu+1}>2n_\nu$, the inequalities

$$\sum_{\nu=0}^{\infty}|c_{n_\nu}|^2<2, \quad 2|c_{n_0}|^2+\sum_{\nu=1}^{\infty}|c_{n_\nu}|^2\leqq 2$$

are sharp, but equality in the first of these is not attainable. A study is made of sets of all points in the plane with the coordinates $(|c_m|, |c_n|)$, where c_m and c_n are the $(m+1)$th and $(n+1)$th coefficients, respectively, in the expansion of some function of class H_1. The cases $n/m>2$ and $2\geqq n/m>\frac{1}{2}$ are investigated and completely solved. The paper concludes with a study of some classes of analytic functions related to the class H_1. *W. Seidel* (Rochester, N. Y.).

Golusin, G. M. On the distortion theorems for "schlicht" conform representation. Rec. Math. [Mat. Sbornik] N.S. **18**(60), 379–390 (1946). (Russian. English summary)

Let S denote the family of functions

$$f(z)=z+c_2z^2+c_3z^3+\cdots$$

which are regular and schlicht in $|z|<1$. R. M. Robinson [Trans. Amer. Math. Soc. **52**, 426–449 (1942); these Rev.

4, 77], using Löwner's parametric representation, characterized the region of variability of the point $(|f(z)|, |f'(z)|)$, z fixed. The author begins by deriving an inequality connecting $|f(z)|$, $|f'(z)|$ and $|z|$ of the type considered by Robinson, and derives from it several precise inequalities involving $|f(z)|$ and $|f'(z)|$ at two different points of $|z|<1$. He also obtains upper bounds for $|f_1(z)|$, $|f_1'(z)|$, $|zf_1'(z)/f_1(z)|$, where $f_1(z)=z+c_3z^3+c_5z^5+\cdots$ is the odd part of $f(z)$. *D. C. Spencer* (Stanford University, Calif.).

*Golusin, G. M. **Interior Problems of the Theory of Schlicht Functions.** Translated by T. C. Doyle, A. C. Schaeffer, and D. C. Spencer. Office of Naval Research, Navy Department, Washington, D. C., 1947. vi+138 pp.

This translation of the Russian survey article by Golusin [Uspekhi Matem. Nauk 6, 26–89 (1939); these Rev. 1, 49] will be welcome to anyone interested in recent research on conformal mapping. The author has set out the main trends of this research (including his own contributions) in great clarity; the translation is very adequate. The following subjects are covered in six chapters. (I) Theorems on distortion and the coefficient problems (the elementary area principle of Bieberbach). (II) The deeper theory based on Löwner's parametric representation of schlicht functions. (III) Radii of convexity and starlikeness. (IV) The strip methods of Grötzsch and Rengel. (V) Schlicht functions which omit a given set of values. (VI) Conformal mapping of multiply-connected regions. Many references to the relevant literature enhance the value of this publication.

W. W. Rogosinski (Newcastle-upon-Tyne).

Spencer, D. C. Some problems in conformal mapping. Bull. Amer. Math. Soc. 53, 417–439 (1947).

This address gives a short expository account of recent methods and results concerning the class of functions $f(z)=z+a_2z^2+a_3z^3+\cdots$, regular and schlicht in $|z|<1$. In particular, the problem of the coefficients is widely discussed. In many respects this account goes beyond the recent survey of this field of research given by Golusin [cf. the preceding review] and it is a valuable complement to it.

W. W. Rogosinski (Newcastle-upon-Tyne).

Schaeffer, A. C., and Spencer, D. C. A general class of problems in conformal mapping. Proc. Nat. Acad. Sci. U. S. A. 33, 185–189 (1947).

Let S denote the class of functions $f(z)=\sum_{\nu=1}^{\infty}a_\nu z^\nu$, $a_1=1$, $|z|<1$, which are regular and schlicht in $|z|<1$. Let R be a closed set within $|z|<1$ and $\psi_\nu(\tau)$ a measure function defined in R. Then $|f^{(\nu)}(z)|\leqq M_n$ for $z\varepsilon R$, $\nu=0, 1, \cdots, n$, for a given n. Let $F_\nu(\zeta_0, \bar{\zeta}_0, \cdots, \zeta_n, \bar{\zeta}_n)$ denote a complex-valued function which with its first derivatives is continuous in an open set containing $|\zeta_\nu|\leqq M_n$, $\nu=0, 1, \cdots, n$. Let

$$P_\nu=\int_R F_\nu(f(z), f'(z), \cdots, f^{(n)}(z), \bar{f}^{(n)}(z))d\psi_\nu, \quad \nu=0, 1, \cdots, m.$$

Then $P_f=(P_1, \cdots, P_m)$, a point in a Euclidean $2m$-space, is said to belong to $f(z)$. As $f(z)$ ranges over S, P_f ranges over a closed bounded set P. What is the region P?

The authors discuss several specific problems included in this general problem. The method applied to these problems can be extended in part to the general problem. One of the specific problems considered is that of finding the region of variability of $f'(z_1)$, $f(z)\varepsilon S$, where z_1 is in $|z|<1$. For a fixed z_1 with $f'(z_1)=p+iq$ the point (p, q) ranges over a closed set P as f ranges over S. Let $G(p, q)$ and its first derivatives be continuous over an open set containing P, with the gradient of G different from zero in P. Let the maximum of G in P occur for a particular $f(z)$ of S. Then $f(z)$ satisfies the differential equation

$$N\left(\frac{zf'(z)}{f(z)}\right)^2\frac{2f(z)-f(z_1)}{\{f(z)-f'(z_1)\}^2}=L\frac{z+z_1}{z-z_1}+L\frac{1+z\bar{z}_1}{1-z\bar{z}_1}+M\frac{2z-z_1}{(z-z_1)^2}+\bar{M}\frac{z(2-z\bar{z}_1)}{(1-z\bar{z}_1)^2},$$

where $N=2G_1f'(z_1)f(z_1)$, $L=G_1z_1f''(z_1)$, $M=2G_1z_1f'(z_1)$, $G_1=\partial G/\partial p-i\partial G/\partial q$, $f'(z_1)=p+iq$. The integrals may be expressed in terms of elementary functions.

The method used for this and other specific problems when applied to the general problem leads to a differential equation for extremal schlicht functions of the form $(dw/dz)^2A(w)=B(z)$, where $A(w)$ and $B(z)$ are rational functions whenever P_ν is of the form

$$P_\nu=\sum_{j=1}^{k}F_\nu(f(z_j), \bar{f}(z_j), \cdots, f^{(n)}(z_j), \bar{f}^{(n)}(z_j)), \quad \nu=1, 2, \cdots, m.$$

M. S. Robertson (New Brunswick, N. J.).

Kufarev, P. On some properties of the kernel function of a domain. Bull. [Izvestiya] Math. Mech. Inst. Univ. Tomsk 3, 72–74 (1946). (Russian)

Let G be a plane domain of finite connectivity, P a boundary point of G and ρ the radius of curvature of the boundary curve at P. The author considers the quantity $H_G(Z)=K_G(Z, \bar{Z})-\pi^{-1}(Z+\bar{Z})^{-2}-2|Z|^2\pi^{-1}\rho^{-1}(Z+\bar{Z})^{-3}$, where $K_G(Z, \bar{T})$ is the kernel function of G. Using the method of the minimum integral [see Bergman, J. Reine Angew. Math. 169, 1–42 (1932)] the author proves that, if Z approaches P along a ray lying inside G, then $\lim_{Z\to P}H_G(Z)$ exists. In the opinion of the reviewer the result can be easily generalized to the case of functions of several complex variables. Since the kernel function is closely related with the Green's function of the domain, the same method of the minimum integral may be applied to the investigation of Robin's constant and the Green's function. It can be extended also to the theory of linear partial differential equations of elliptic type.

S. Bergman (Cambridge, Mass.).

Heins, Maurice. The minimum modulus of a bounded analytic function. Duke Math. J. 14, 179–215 (1947).

Soit $f(z)$ une fonction analytique définie dans le cercle $|z|<1$ et dont le module y est inférieur à 1. Le module minimum de $f(z)$ étant $m(r; f)=\min_{(\theta)}|f(re^{i\theta})|$ (r et θ réels, $0\leqq r<1$), l'auteur considère la fonction $m^*(r; f)$ définie par $m^*(r; f)=\sup_{r\leq\rho<1}m(\rho; f)$. Dans l'étude de la croissance de $m^*(r; f)$ un intérêt particulier s'attache à la notion de "fonction dominée exponentiellement" (d.e.). Une fonction de la classe considérée est dite d.e., s'il existe un α positif et un θ_0 réel tels que pour $|z|<1$ on ait

$$|f(z)|\leqq\exp\left\{-\alpha\frac{1-|z|^2}{|e^{i\theta_0}-z|^2}\right\}.$$

Le théorème principal démontre alors: (1) l'existence de $\lim_{r\to1}(1-r)\log m^*(r; f)$ dans tous les cas; (2) cette limite est $-\infty$ seulement si $f(z)\equiv0$; (3) elle est égale à 0 seulement si $f(z)$ n'est pas d.e.; (4) elle est finie et négative seulement si $f(z)$ n'est pas la constante 0 et est d.e. La valeur exacte de la limite dans ce dernier cas peut être déterminée à l'aide d'une expression, qui découle de la formule de Poisson-Stieltjes pour la représentation des fonctions bornées. L'étude

des fonctions extrémales (quant à la croissance de $m^*(r;f)$ et de $m(r;f)$) de la classe considérée montre le rôle prépondérant que jouent les fonctions représentées par des produits de Blaschke. *B. Amirâ* (Jérusalem).

Shah, S. M. A note on the minimum modulus of a class of integral functions. Bull. Amer. Math. Soc. **53**, 524–529 (1947).

The author proves results concerning the density of the set of r for which $\log m(r) > C \log M(r)$, where $m(r)$ and $M(r)$ are the minimum modulus and maximum modulus of an entire function $f(z)$ whose zeros are restricted to lie in certain annular regions of rapidly increasing radii. The results are established first for a canonical product.

R. C. Buck (Providence, R. I.).

Valiron, Georges. Généralités sur les fonctions entières. Revista Unión Mat. Argentina **12**, 105–117 (1947).

Lecture at the University of Buenos Aires.

Bernstein, Serge. Démonstration nouvelle et généralisation de quelques formules de la meilleure approximation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 663–664 (1946).

The author uses theorem 7 of his fifth note on best approximation [same vol., 475–478 (1946); these Rev. **8**, 509] to complete previous results. If $A_p(f)$ is the best approximation of $f(x)$ in $(-\infty, \infty)$ by entire functions of exponential type p, then $A_p\{(x^2+a^2)^s\} = 2^s a^{s-1} e^{-pa} p^{-s-1} / |\Gamma(-s)|$, $-\infty < s < \infty$. He then proves that, if $f(x)$ is continuous $(a \leqq x \leqq b)$ and $f^{(k)}(x)$ and $f^{(l)}(x)$ uniformly continuous for $x \geqq b$ and $x \leqq a$, respectively (k, l nonnegative integers), then $A_p(f) \to 0$ as $p \to \infty$. Finally he deduces a result on $E_n\{|x|^s(\log|x|)^m\}$, where $s>0$, m is a positive integer, $E_n(f)$ is the best approximation of $f(x)$ by polynomials of degree n in $(-1, 1)$. *H. Kober* (Birmingham).

Selberg, Henrik L. Über einen Satz von Collingwood. Arch. Math. Naturvid. **48**, no. 9, 119–126 (1944).

Selberg, Henrik L. Über den zweiten Hauptsatz der Wertverteilungslehre. Arch. Math. Naturvid. **48**, no. 10, 127–144 (1944).

The author has returned to the subject matter of these papers in a later publication [Comment. Math. Helv. **18**, 309–326 (1946); these Rev. **8**, 23] in which the same and sharper results are derived. *L. Ahlfors.*

Gerst, Irving. Meromorphic functions with simultaneous multiplication and addition theorems. Trans. Amer. Math. Soc. **61**, 469–481 (1947).

Le but du travail présent est de déterminer toutes les fonctions méromorphes satisfaisant simultanément aux équations de la forme $f(ms) = R(f(s))$, $f(s+h) = S(f(s))$, où R et S désignent des fonctions rationnelles données. En s'appuyant aux résultats de Ritt [Trans. Amer. Math. Soc. **23**, 16–25 (1922)] concernant les fonctions périodiques ayant un théorème de multiplication rationnel l'auteur démontre que les fonctions en question se réduisent en général aux fonctions linéaires, exponentielles ou elliptiques.

P. J. Myrberg (Helsingfors).

Yang, T. Analyse zur Definition der Riemannschen Flaechen. Tôhoku Math. J. **49**, 208–212 (1943).

General remarks concerning the definition of Riemann surfaces and covering surfaces. No new ideas are presented.

L. Ahlfors (Cambridge, Mass.).

Myrberg, P. J. Über einige Probleme der Funktionentheorie. S.-B. Finnisch. Akad. Wiss. **1945**, 229–233 (1946).

The author gives a brief account in historical setting of his recent investigations concerning the extension of the notion of Abelian integrals to open Riemann surfaces and related problems. *M. Heins* (Providence, R. I.).

Araviyskaja, E. N. Sur la théorie de la croissance des fonctions dont les variétés-zéro sont données. Bull. [Izvestiya] Math. Mech. Inst. Univ. Tomsk **3**, 61–71 (1946). (Russian. French summary)

The author considers the inequality

$$|\log|f(z_1, z_2)||^2 \leqq AH(z_1, z_2)$$

for functions of two complex variables which are meromorphic in a domain B of the space of two complex variables z_1, z_2, $z_k = x_k + iy_k$. Here $A = \int_B (\log|f|)^2 d\omega$, $d\omega = dx_1 dy_1 dx_2 dy_2$, and $H(z_1, z_2)$ is the kernel function for the biharmonic functions with respect to the domain B', B' being the domain obtained from B by deleting the zero- and pole-surfaces of f [see Bergman, Math. Ann. **109**, 324–348 (1934)]. Using the property that the kernel function increases if the domain decreases, the author gives an upper bound for $H(z_1, z_2)$; namely, she determines the kernel function of the domain $R: |z_2|^2 + a|z_1|^{2/p} \leqq 1$ and proves that a pseudoconformal image C of R (assuming a suitable choice of a and p) can be placed in the interior of B'. Since $H_C(z_1, z_2) \geqq H(z_1, z_2)$ and the kernel function of H_C can be represented in closed form, the author obtains the upper bound.

The reviewer remarks that the considerations of the author could be applied to the inequality

$$|\log|f(z_1, z_2)|| \leqq (\prod Z_\nu / \prod P_\nu)\{\chi(z_1, z_2) D\}^{\frac{1}{2}},$$

$$D = \int_B (\log|f|)^2 d\omega - \int_B (\log|\prod Z_\nu / \prod P_\nu|)^2 d\omega,$$

where χ is the kernel-function (for biharmonic functions) of B and P_ν and Z_ν are the normalized pole- and zero-functions, respectively, of f with respect to B. *S. Bergman.*

Fourier Series and Generalizations, Integral Transforms

Alaci, V. Une classe d'intégrales définies, nécessaire dans la théorie des séries trigonométriques. Bull. Sci. École Polytech. Timişoara **12**, 129–141 (1946).

From an explicit determination of the coefficients of the polynomial $(1-x^{2n})^p(1-x)^r/(1-x^2)^q$, where p, q, n and r are positive integers and $q \leqq p$, trigonometric identities are obtained for the expression $\sin^p 3nt \sin^q t / \sin^q 3t$. The values of the integrals over $(0, \pi)$ and $(0, \pi/2)$ of the latter quantities are determined and for integrable $f(x)$ the integral $(1/\pi)\int_0^\pi (\sin^p 3nt \sin^q t / \sin^q 3t) f(x+t)dt$ is evaluated in terms of the partial sums of the Fourier series of $f(x)$.

P. Civin (Eugene, Ore.).

Sargent, W. L. C. On the order of magnitude of the Fourier coefficients of a function integrable in the $C_\lambda L$ sense. J. London Math. Soc. **21**, 198–203 (1946).

The Fourier coefficients of a function integrable $C_\lambda L$ are $o(n^{\lambda+1})$, and this is the best possible estimate [for the definition of integrability $C_\lambda L$, see Bosanquet, Proc. London Math. Soc. (2) **49**, 40–62 (1945); these Rev. **7**, 280].

A. Zygmund (Chicago, Ill.).

Menchoff, D. Sur les sommes partielles des séries trigonométriques. Rec. Math. [Mat. Sbornik] N.S. **20(62)**, 197–238 (1947). (Russian. French summary)

Complete proofs of results announced previously [C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 79–82 (1945); these Rev. **7**, 435]. *A. Zygmund* (Chicago, Ill.).

Misra, M. L. On the non-summability $(C, 1)$ of Fourier series. Bull. Amer. Math. Soc. **53**, 510–514 (1947).

An example of Prasad [Math. Z. **40**, 496–502 (1935)] shows that if (*) $f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ and $\varphi(t)=f(x+t)+f(x-t)-2f(x)$, the convergence of $\int_t^\pi\varphi(u)u^{-1}du$ as $t\to0$ is not sufficient for $(C, 1)$ summability of (*) at the point x. The present paper shows that condition $\int_t^\pi\varphi(u)u^{-1}du-s=o(\log\log(1/t))$ as $t\to0$, for some s, is still insufficient for $(C, 1)$ summability at x. In a slightly different vein, Hsiang has recently shown [Duke Math. J. **13**, 43–50 (1946); these Rev. **7**, 293] that the convergence of $\int_t^\pi\varphi(u)u^{-1-\eta}du$ for any $\eta>0$ as $t\to0$ will imply the $(C, 1)$ summability of (*) at the point x. *P. Civin.*

Misra, M. L. On the Cesàro summability of the successively derived series of the conjugate series of a Fourier series. Bull. Calcutta Math. Soc. **38**, 151–155 (1946).

The author shows that the rth derived conjugate Fourier series is summable $(C, \alpha+r)$ if a certain function $g(t)$ is integrable L and its conjugate series with respect to the range $(-\pi, \pi)$ is summable (C, α) for $t=0$. [The case $r=1$ is included in a theorem of T. Takahashi and F. T. Wang, Tôhoku Math. J. **41**, 169–187 (1935), theorem 3.] He states incorrectly that the sum of the first series is $r!$ times that of the second. The right sums are

$$r!(2/\pi)\int_0^\infty t^{-1}g(t)dt,\quad (2/\pi)\int_0^\infty t^{-1}g_1(t)dt,$$

where $g_1(t)$ is $g(t)$ altered so as to be periodic. The reviewer has shown that the condition of integrability L may be replaced by integrability in a more general sense, and that the conditions are then necessary [Proc. London Math. Soc. (2) **49**, 63–76 (1945), theorem 1 and lemma 10; these Rev. **7**, 154]. *L. S. Bosanquet* (London).

Misra, M. L. The summability (A) of the successively derived series of a Fourier series and its conjugate series. Duke Math. J. **14**, 167–177 (1947).

The author obtains a sufficient condition for the summability (A) of the rth derived Fourier series. [This is a corollary of a theorem of N. Obreschkoff, Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1941**, no. 15 (1942), theorem 1, with $\alpha=1$; these Rev. **8**, 262, which was presumably unknown to the author.] He also gives a sufficient condition for the summability (A) of the rth derived conjugate series. [This is a particular case of a theorem of A. H. Moursund, Duke Math. J. **2**, 485–491 (1936), theorem 1, with $p=0$ and $\alpha_{r-1-2\nu}=0$ for $\nu<\frac{1}{2}(r-1)$.] *L. S. Bosanquet* (London).

Natanson, I. P. Sur la répresentation approchée des fonctions vérifiant la condition de Lipschitz au moyen de l'intégrale de Vallée-Poussin. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 11–13 (1946).

Let $f(x)$ be a function of period 2π satisfying the condition $|f(y)-f(x)|\leqq|y-x|^\alpha$ $(0<\alpha\leqq1)$ and let

$$V_n[f; x]=\frac{1}{\pi}\,\frac{2\cdot4\cdots2n}{1\cdot3\cdots(2n-1)}\int_{-\pi}^{\pi}f(t)\cos^{2n}\tfrac{1}{2}(t-x)dt.$$

Let furthermore $\Delta_n(f)=\max_x|V_n[f; x]-f(x)|$ and $U_n(\alpha)=\text{l.u.b.}_f\,\Delta_n(f)$. The author proves that

$$U_n(\alpha)=(2^\alpha/\pi)^{\frac{1}{2}}\Gamma((1+\alpha)/2)n^{-\alpha/2}+o(n^{-\alpha/2}),\quad 0<\alpha\leqq1;$$
$$U_n(1)=2/(n\pi)^{\frac{1}{2}}+n^{-\frac{1}{2}}\theta_n,\quad -1<\theta_n<1.$$

[In the integral at the bottom of p. 11, f should be replaced by t. There are several other minor misprints.]

M. Kac (Ithaca, N. Y.).

Sansone, G. Sull'approssimazione di funzioni continue con polinomi trigonometrici. Boll. Un. Mat. Ital. (3) **1**, 39–42 (1946).

The well-known result of D. Jackson concerning the degree of approximation of functions of period 2π by trigonometric polynomials [see his "The Theory of Approximation," Amer. Math. Soc. Colloquium Publ., v. 11, New York, 1930, p. 12] is extended here, with proper modifications, to the case of functions defined in the interval $[0, \pi]$ only. Thus, let $f(x)$, $0\leqq x\leqq\pi$, have a continuous derivative $f^{(2p)}(x)$ of even order and let $\omega_{2p}(\delta)$ be the modulus of continuity of $f^{(2p)}$. Then for each $n\geqq p$ there is a trigonometric polynomial T_n of order n such that

$$|f-T_n|\leqq A_p\left[\omega_{2p}(2\pi/n)+\left(\sum_{k=1}^{p}|f^{(2k-1)}(0)|+|f^{(2k-1)}(\pi)|\right)n^{-1}\right]n^{-2p}$$

with A_p depending on p only. An analogous result holds for functions differentiable an odd number of times.

A. Zygmund (Chicago, Ill.).

Civin, Paul. Mean values of periodic functions. Bull. Amer. Math. Soc. **53**, 530–535 (1947).

Let L^p denote the class of complex measurable functions of period 2π for which $M_p(f)<\infty$, where

$$M_p(f)=\left(\int_0^{2\pi}|f(x)|^p dx\right)^{1/p},\qquad 1\leqq p<\infty.$$

Let $K_{m,p}$ denote the subset of L^p whose elements $f(x)$ have a Fourier series of the form

$$(*)\qquad \sum_{n=m}^{\infty}(a_n\cos nx+b_n\sin nx),\qquad m\geqq1.$$

Consider a transformation of (*) into

$$(**)\qquad \sum_{n=m}^{\infty}\lambda(n)\{a_n\cos(nx+\delta\pi/2)+b_n\sin(nx+\delta\pi/2)\},$$

where δ, $\lambda(n)$ are real numbers. Then, under the assumption that $\sum_{n=m}^{\infty}\lambda(n)\cos(nx-\delta\pi/2)$ is a Fourier series, it follows that (**) is again a Fourier series of a function $g(x)\in K_{m,p}$. Thus there is a transformation of $K_{m,p}$ onto itself, and the author proves an inequality between the means $M_p(f)$ and $M_p(g)$, which holds for all $f(x)\in K_{m,p}$. This inequality is an extension of a result of B. v. Sz. Nagy [Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl. **90**, 103–134 (1938)], who considered only the case of $K_{m,\infty}$.

K. Chandrasekharan (Princeton, N. J.).

Edmonds, Sheila M. The Parseval formulae for monotonic functions. I. Proc. Cambridge Philos. Soc. **43**, 289–306 (1947).

Let $f(t)$, $g(t)$ be defined for $0\leqq t<\infty$ and let F_c, G_c, F_s, G_s be their cosine and sine transforms. If $f, g\in L^2(0, \infty)$, the equations

$$(*)\qquad \int_0^\infty F_cG_c\,dx=\int_0^\infty fg\,dt,\quad \int_0^\infty F_sG_s\,dx=\int_0^\infty fg\,dt$$

are only special cases of the general Parseval-Plancherel formula. In a previous paper [same Proc. **38**, 1–19 (1942); these Rev. **3**, 231] the author investigated the validity of these formulas under different assumptions and, in particular, proved that they hold if f and g decrease monotonically to 0 in $(0, \infty)$ and if $fg \in L(0, \infty)$. Whether this last assumption can be dropped is the main problem of the present paper; the answer is on the whole affirmative. The following results may serve as examples. (i) If f and g decrease to 0 in $(0, \infty)$, the second formula (*) holds (even if either side is infinite). (ii) If f and g decrease to 0 and are convex in $(0, \infty)$, the first formula (*) holds. The results remain valid if we replace the expression "f and g" by "f and G_ϵ" in (i) and by "f and G_ϵ" in (ii). *A. Zygmund* (Chicago, Ill.).

Ramlau, P. N. On the frequency spectra of repeated signals. Avtomatika i Telemehanika **1941**, no. 1, 33–42 (1941). (Russian)

The author shows the simplifications that occur in the formulas involving the Fourier transform of a function $F(t)$ when $F(t)$ has period T_1 when $0 \leqq t \leqq (n-1)T_1$ and $F(t)$ vanishes outside of $(0, nT_1)$. *N. Levinson.*

Boas, R. P., Jr., and Pollard, Harry. Complete sets of Bessel and Legendre functions. Ann. of Math. (2) **48**, 366–384 (1947).

This paper is concerned with $L_p(E)$ completeness of sequences of Bessel functions and of generalized Legendre polynomials. Here E is some finite interval and $p = \infty$ is not excepted. Completeness with the addition of just one other function is also considered. For the Legendre polynomials a connection is established between $L_p(a, b)$ completeness and $B(p)$ completeness of an associated class of trigonometric functions on a correlated interval where $B(p)$ is a function class determined by p and a and b. This is a natural conjugation in view of the classical relations between Sturm-Liouville and cosine series but the reader will appreciate the results obtained from the fact that the asymptotic character of the Sturm-Liouville functions central in the classical reasoning has no counterpart for the results here when for instance $-1 < a < b < 1$. In loose analogy to the results of Müntz and Szász on $\{x^{\lambda_n}\}$ it appears that the values of a and b profoundly affect the results obtained though in this work the question is one of inclusion of 1 in the subinterval $-1 \leqq a < b \leqq 1$. Typical theorems are: $\{x^{\frac{1}{2}} J_\nu(\lambda_n x)\}$ is $L_p(0, 1)$ complete for $p > (1-\delta)^{-1}$ if $\nu > -\frac{1}{2}$, $0 \leqq \delta < 1$ and λ_n is the nth positive root of $J_{\nu+1}(\lambda)$; $\{P_{\lambda_n}(x)\}$ is $L_p(a, 1)$ complete, $p > 2$, if $\{\cos(\lambda_n + \frac{1}{2})\}$ is complete Lip α, $\alpha \leqq (p-2)p^{-1}$, where Lip α characterizes a function satisfying a Lipschitz condition of degree α on $(0, \cos^{-1} a)$. The methods are of the type pursued by Wiener and by Levinson. Thus an entire function (similar to $\sin z$) is defined by a product representation with zeros at $z = \lambda_n$. Then the Poisson-Jensen theorem and theorems analogous to the Lerch or Carlson theorems, etc., are used to show that the order or the vanishing of a transform of a function on a set implies that the function vanishes almost everywhere. The transforms used come from the integral representations of the Bessel functions or generalized Legendre polynomials.

D. G. Bourgin (Urbana, Ill.).

Pollard, Harry. The inversion of the transforms with reiterated Stieltjes kernels. Duke Math. J. **14**, 129–142 (1947).

Der Verfasser beweist in Verallgemeinerung früherer Resultate von R. P. Boas und D. V. Widder den folgenden Umkehrungssatz. Es bedeute $H_1(x, y)$ den Kern $(x+y)^{-1}$. Die iterierten Kerne $H_n(x, y)$ haben die Werte

$$H_n(x, y) = \int_0^\infty \frac{H_{n-1}(x, y)}{u+y} du, \quad n = 2, 3, \cdots.$$

Es sei

$$f(x) = \int_{0+}^{\infty} H_n(x, y) d\alpha(y) = \lim_{R \to \infty,\ \epsilon \to 0+} \int_\epsilon^R H_n(x, y) d\alpha(y),$$

wobei $\alpha(y)$ in jedem Intervall (ϵ, R) von beschränkter Variation ist. Unter den Operatoren $L_{k,x}(f)$, $k = 0, 1, 2, \cdots$, verstehen wir die folgenden Funktionen:

$$L_{k,x}(f) = \{\Gamma(k+\tfrac{1}{2})\}^{-2} (-x)^k [x^k f(x)]^{(2k)}.$$

Dann gilt die Umkehrformel

$$\lim_{h \to \infty} \int_{0+}^{x} L_{h,t}^n(f) dt = \alpha(x) - \alpha(0+)$$

in allen Stetigkeitspunkten von $\alpha(x)$. *W. Saxer.*

Haviland, E. K. A note on Laplace transforms of functions whose spectra are confined to a given set. Amer. J. Math. **69**, 279–285 (1947).

Der Verfasser beweist den folgenden Satz. Notwendig und hinreichend für die Darstellbarkeit der Funktion $f(x)$ in der Form $f(x) = \int_0^\infty e^{-xu} d\beta(u)$, wobei $\beta(u)$ eine beschränkte, nicht abnehmende Funktion mit einer bestimmten Teilmenge Γ von $0 \leqq x < +\infty$ als Spektrum ist und das Integral für jeden Wert x dieses Intervalles konvergiert, sind die folgenden Bedingungen: $f(x)$ muss in diesem Intervall vollkommen monoton sein und wenn $a_0 + a_1 e^{-u} + \cdots + a_n e^{-nu}$ irgend ein nicht negatives Exponentialpolynom auf Γ bedeutet, muss $a_0 f(x) + a_1 f(x+1) + \cdots + a_n f(x+n) \geqq 0$ für alle $x \geqq 0$. Bezeichnungen: (1) Unter dem Spektrum der Funktion $\beta(t)$ wird die Menge aller Punkte τ bezeichnet, sodass $\beta(t_2) - \beta(t_1) > 0$ für jedes Intervall $t_1 < \tau < t_2$. (2) Eine Funktion $f(x)$ gilt als vollkommen monoton im Intervall $0 < x < \infty$, wenn die $p$te Ableitung für alle positiven x der Bedingung genügt $(-1)^p f^{(p)}(x) \geqq 0$, $p = 0, 1, 2, \cdots$, und $f(x)$ für $x = 0$ stetig ist, sofern x auch den Wert 0 annehmen kann.

W. Saxer (Zürich).

Labin, Eduardo. Operational calculus. Ciencia y Técnica **108**, 1–11, 43–56, 96–108, 159–170, 209–231 (1947). (Spanish. French summary)

Expository article.

***Jessen, Børge. Mouvement moyen et distribution des valeurs des fonctions presque-périodiques.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 301–312. Jul. Gjellerups Forlag, Copenhagen, 1947.

This is primarily an expository paper showing how the problem of Lagrange led to some of its modern generalizations connected with the title of the paper. In the closing section unpublished results of the author in collaboration with Borchsenius are mentioned. These results give more precise information concerning the distribution of the values of the Riemann zeta function in the half plane $\frac{1}{2} < \Re z$.

R. H. Cameron (Minneapolis, Minn.).

***Tornehave, Hans. On the variation of the argument of exponential polynomials in two variables.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 325–328. Jul. Gjellerups Forlag, Copenhagen, 1947.

If $P(x, y) \equiv \sum_{j=1}^n a_j \exp i(h_j x + k_j y)$ never vanishes as x, y range over all real number pairs (h_j, k_j being integers), and

if $\arg P-px-qy$ is continuous and of period 2π in both variables, then (p, q) is called the mean motion vector of P. The author gives results concerning the location of mean motion vectors and, in particular, shows that if P has at most five exponent vectors (h_j, k_j) and no three are collinear, then the mean motion vector must be one of the exponents. *R. H. Cameron* (Minneapolis, Minn.).

Martchenko, V. A. Application de la méthode de sommation de Fejer-Bochner aux séries de Fourier généralisées. C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 7–9 (1946).

The author considers measurable complex functions $f(x)$ of the real variable x such that

$$\limsup_{T\to\infty}(2T)^{-1}\int_{-T}^{T}|f_n(x)|dx$$

is bounded in n, where $f_n(x)=f(x)\min[1, n|f(x)|^{-1}]$. Using the Lewitan mean

$$M\{f(t)\}=\lim_{n\to\infty}\left\{\operatorname{Lim}_{T\to\infty}(2T)^{-1}\int_{-T}^{T}f_n(x)dx\right\}$$

(in which Lim denotes the Banach limit), the author defines the Fourier coefficients $a(\lambda)=M\{f(t)\exp(-i\lambda t)\}$ and states that $a(\lambda)$ is nonvanishing for only a countable set of λ, so that each of his functions has a generalized Fourier series $f(x)\sim a(0)+\sum a(\lambda_k)\exp i\lambda_k x$. He now defines a topology on the real axis in terms of which "continuous" and "uniformly continuous" functions are necessarily Lewitan almost periodic and Bohr almost periodic functions, respectively, and functions for which $a(\lambda)\equiv 0$ vanish at every point of "continuity." Other theorems concerning the approximation of the author's functions and summability of his Fourier series are stated. In particular, if $f(x)$ is bounded, its Fourier series is summable to $f(x)$ by the Bochner-Fejér method at each point of "continuity" of $f(x)$. *R. H. Cameron.*

Polynomials, Polynomial Approximations

Sherman, Seymour. Generalized Routh-Hurwitz discriminant. (An extension of the theorems of Sturm, Routh, and Hurwitz on the roots of polynomial equations.) Philos. Mag. (7) **37**, 537–551 (1946).

The Sturm sequence is constructed for the number of zeros of a complex polynomial in a sector by use of the Cauchy index theorem. Also the Routh algorithm and Hurwitz determinant criterion are extended to give the number of zeros of a complex polynomial in a half-plane bounded by any line through the origin. [Reviewer's note. On the construction of Sturm sequences for a region, reference should be given to Serret's Cours d'Algèbre Supérieure, 7th ed., Paris, 1928, v. 1, pp. 292–297. For the extension of the Hurwitz criterion to complex polynomials, reference should be given to H. Bilharz, Z. Angew. Math. Mech. **24**, 77–82 (1944); these Rev. **7**, 62, and to E. Frank, Bull. Amer. Math. Soc. **52**, 144–157 (1946); these Rev. **7**, 295.] *M. Marden* (Milwaukee, Wis.).

Viola, Tullio. Sulle equazioni algebriche a coefficienti reali, le cui radici hanno parti reali esterne o non interne a un determinato intervallo. Rend. Accad. Sci. Fis. Mat. Napoli (4) **9**, 15–20 (1939).

Necessary and sufficient conditions for all the roots of an algebraic equation having only real coefficients to have their real parts positive are well known [cf. A. Hurwitz, Math. Ann. **46**, 273–284 (1895)]. Here conditions for such an equation to have the real parts of its roots outside (not inside) a given interval are found; these conditions are that certain polynomials must be positive definite (semidefinite). *O. Todd-Taussky* (Washington, D. C.).

Best, G. C. The determination of the complex zeros of a polynomial. Amer. Math. Monthly **54**, 269–273 (1947).

The author describes a method for determining the real part $-\alpha/2$ of a complex root of a polynomial $P(x)$ if its modulus $r=\beta^{\frac{1}{2}}$ is known. Since $P(x)=a_0+\cdots+a_nx^n$ must be divisible by $\beta+\alpha x+x^2$ or $1-ux-kx^2$, where $u=-\alpha/\beta$, $k=-1/\beta$, the series for $P(x)/(1-ux-kx^2)=\sum p_ix^i$, where $p_i=up_{i-1}+kp_{i-2}+a_i$, must have $p_{n-1}(u)=p_n(u)=0$. Then the greatest common divisor of $p_{n-1}(u)$, $p_n(u)$ must vanish. Its roots $u_1, \cdots, u_s$ yield s values of α and $2s$ values of x. Explicit formulae are given for $n=3, 4, 5$. *E. Bodewig.*

Lewis, D. C. Polynomial least square approximations. Amer. J. Math. **69**, 273–278 (1947).

Expression sous forme intégrale de la différence $f(x)-P_n(x)$, où $P_n(x)$ désigne un polynome, de degré non supérieur à n, tel que la quantité

$$\sum_{k=0}^{p}\int_a^b\{f^{(k)}(x)-P_n^{(k)}(x)\}^2d\alpha_k(x)$$

soit minimum; les $\alpha_k(x)$ désignent des fonctions non décroissantes de x ayant un nombre suffisant de points de croissance et la fonction $f(x)$ est telle que le problème précédent ait une solution unique. *J. Favard* (Paris).

Nikolsky, S. On the best approximation in the mean to the function $|a-x|^s$ by polynomials. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **11**, 139–180 (1947). (Russian. English summary)

Bernstein [C. R. (Doklady) Acad. Sci. URSS (N.S.) **18**, 379–384 (1938)] investigated the best approximation of the function $|a-x|^s$ by polynomials. In the present paper the author studies the best approximation of $|a-x|^s$ in the mean. Let

$$E_{n,r(x)}[f]=\min_{P_n}\int_{-1}^{1}|f(x)-P_n(x)|r(x)dx,$$

where the minimum is taken for all the polynomials P_n of degree n, and $r(x)$ is the weight function. Let $E_n[f]=E_{n,1}[f]$. It is shown, among other things, that (i) if $-1<a<1$ and if s exceeds -1 and is not an even integer, then

$$E_n[|x-a|^s]=M_s(1-a^2)^{\frac{1}{2}(s+1)}n^{-s-1}+O(n^{-s-2}\log n)$$

with M_s depending on s only; (ii) if $s>-1$, $-1<a<1$, then asymptotically, for $n\to\infty$,

$$E_{n,r(x)}[|x-a|^s]\cong r(a)E_n[|x-a|^s];$$

(iii) if $s>-1$, $-1<a_1<\cdots<a_m<1$, $f(x)=\sum_1^m A_k|x-a_k|^s$, then

$$E_{n,r(x)}[f]\cong M_sn^{-s-1}\sum_1^m|A_k|(1-a_k^2)^{\frac{1}{2}(s+1)}r(a_k).$$

A. Zygmund (Chicago, Ill.).

Schramm, Reuben. A new proof for the approximation theorem of Weierstrass. Riveon Lematematika **1**, 50–53 (1946). (Hebrew)

The continuous function is approximated by a step function, the step function is a sum of unit functions such as

$\frac{1}{2}(1+y/|y|)$ which has the value unity for positive y and the value zero for negative y, and each unit function is approximated by a partial sum of the series

$$\frac{1}{2}\left\{1+\frac{y}{t}\sum\binom{-\frac{1}{2}}{n}\left(\frac{y^2}{t^2}-1\right)^n\right\}, \qquad t>0.$$

A. Erdélyi (Pasadena, Calif.).

Simonart, Fernand. Sur les systèmes de polynomes de Sturm-Liouville. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 8–21 (1947).

This article does not contain new results. It deals with the classification of Sturm-Liouville polynomials established by S. Bochner [Math. Z. **29**, 730–736 (1929)].

E. Kogbetliantz (New York, N. Y.).

Special Functions

Bose, S. K. Integrals involving Legendre and Bessel functions. Bull. Calcutta Math. Soc. **38**, 181–184 (1946).

Evaluation of the integrals

$$I_m=\int_0^1 P_n(x)\prod_{i=1}^{m}J_{p_i}(2xy)x^{r-\frac{1}{2}}dx,$$

$$I'_m=\int_0^1 P_n(1-2x^2)\prod_{i=1}^{m}J_{p_i}(xy)x^{r-\frac{1}{2}}dx.$$

A. Erdélyi (Pasadena, Calif.).

Bose, S. K. On certain integrals involving Legendre and Bessel functions. Bull. Calcutta Math. Soc. **38**, 177–180 (1946).

The integrals I_1' [for notation cf. the preceding review],

$$\int_0^1 (1-2xy+y^2)^{-\frac{1}{2}}J_m(2px)x^{r-\frac{1}{2}}dx,$$

and

$$\int_0^1 (1-2xy+y^2)^{-\frac{1}{2}}J_m(2px)J_n(2px)x^{r-\frac{1}{2}}dx$$

are evaluated by expanding the integrand in powers of y and integrating term by term, and the integral

$$\int_0^1 \{(1-h)^2+4hy^2\}^{-\frac{1}{2}}\,\mathrm{bei}\,\{2(2xy)^{\frac{1}{2}}\}dy$$

is evaluated by operational methods. *A. Erdélyi.*

Infeld, L., Smith, V. G., and Chien, W. Z. On some series of Bessel functions. J. Math. Phys. Mass. Inst. Tech. **26**, 22–28 (1947).

Being confronted with the series $\sum_{m=0}^{\infty}(-1)^m Y_0(mx)$ in a study of radiation from a cylindrical antenna in a rectangular wave guide, the authors consider the more general problem of finding

$$S=2\pi\sum_{m=1}^{\infty}\cos(2\pi km)H_0^{(2)}(2\pi z\sqrt{m^2+\epsilon^2}), \qquad 0\leqq k<1.$$

By means of the lemmas that

$$(2/\pi)^{\frac{1}{2}}\int_0^{\infty}(x^2-z^2)^{-\frac{1}{2}}e^{-\epsilon(x^2-z^2)^{\frac{1}{2}}}\cos(tx)dx$$

$$=(\pi/2)^{\frac{1}{2}}e^{-\frac{1}{2}i\pi}H_0^{(2)}(z(t^2+\epsilon^2)^{\frac{1}{2}})$$

and that the Fourier transform of

$$g(t)=\cos(2\pi kt)H_0^{(2)}(z(t^2+\epsilon^2)^{\frac{1}{2}})$$

is

$$f(x)=\frac{i}{(2\pi)^{\frac{1}{2}}}\left[\frac{\exp(-\epsilon\{(x+2k\pi)^2-z^2\}^{\frac{1}{2}})}{\{(x+2k\pi)^2-z^2\}^{\frac{1}{2}}}+\frac{\exp(-\epsilon\{(x-2k\pi)^2-z^2\}^{\frac{1}{2}})}{\{(x-2k\pi)^2-z^2\}^{\frac{1}{2}}}\right]$$

the required summation formula is found to be

$$S=i\frac{\exp(-2\pi\epsilon(k^2-z^2)^{\frac{1}{2}})}{(k^2-z^2)^{\frac{1}{2}}}+i\sum_{n=1}^{\infty}\left[\frac{\exp(-2\pi\epsilon\{(n+k)^2-z^2\}^{\frac{1}{2}})}{\{(n+k)^2-z^2\}^{\frac{1}{2}}}+\frac{\exp(-2\pi\epsilon\{(n-k)^2-z^2\}^{\frac{1}{2}})}{\{(n-k)^2-z^2\}^{\frac{1}{2}}}\right]-\pi H_0^{(2)}(2\pi\epsilon z).$$

The (not trivial) transition $\epsilon\to 0$ gives

$$2\pi\sum_{m=1}^{\infty}\cos(2\pi km)H_0^{(2)}(2\pi zm)=-\pi+2i\log\tfrac{1}{2}z-i\left[\frac{d}{dk}\log\Gamma(1+k)-\frac{d}{dk}\log\Gamma(1-k)\right]+\frac{i}{(k^2-z^2)^{\frac{1}{2}}}+i\sum_{n=1}^{\infty}\left[\left(\frac{1}{\{(n+k)^2-z^2\}^{\frac{1}{2}}}-\frac{1}{n+k}\right)+\left(\frac{1}{\{(n-k)^2-z^2\}^{\frac{1}{2}}}-\frac{1}{n-k}\right)\right].$$

Numerous special cases, for the most part well-known, are stated. A table for $\sum_{m=1}^{\infty}(-1)^m Y_0(m\pi x)$ is computed for $x=0.0(0.2)3.0$. To avoid the difficulty of the slow convergence the series on the right is broken into two parts and the radicals in the second part are expanded.

S. C. van Veen (Delft).

Blanuša, Danilo. Eine Klasse von Integraltheoremen der Besselschen Funktionen. Bull. Intern. Acad. Croate. Cl. Sci. Math. Nat. **35**, 103–139 (1945).

In part I [A boundary problem as the origin of certain integral theorems] the equation of telegraphy

$$\frac{\partial^2 U}{\partial x^2}=\frac{1}{\gamma^2}\frac{\partial^2 U}{\partial t^2}+\frac{2\rho}{\gamma^2}\frac{\partial U}{\partial t}+\frac{\rho^2-\sigma^2}{\nu^2}U$$

is transformed into $\partial^2 z/\partial\xi\partial\eta=z$, $z=Ue^{\rho t}$, which satisfies the boundary conditions $z(\xi,\xi)=0$ for $\xi<0$, $z(\xi,\xi)=\varphi(\xi)$ for $\xi\geqq 0$ ($\varphi(\xi)$ of class C'' in $(0,\infty)$), $z(\xi,\eta)=0$ for $\xi+\eta<0$, $\xi-\eta\geqq 0$. The solution of Doetsch [Festschr. Tech. Hochsch. Stuttgart, Springer, Berlin, 1929, pp. 56–70], which is not uniquely determined, is discussed. By the method of Riemann the author obtains the solution

$$z(\xi,\eta)=\varphi(\eta)+(\xi-\eta)\int_0^{\eta}\varphi(x)\frac{J_1(2(\xi-x)^{\frac{1}{2}}(x-\eta)^{\frac{1}{2}})}{(\xi-x)^{\frac{1}{2}}(x-\eta)^{\frac{1}{2}}}dx.$$

As a by-product the following integrals are obtained and stated as new:

$$\int_a^t\frac{J_1(t-y)}{t-y}J_0\{(y^2-a^2)^{\frac{1}{2}}\}dy=(t-a)\frac{J_1((t^2-a^2)^{\frac{1}{2}})}{(t^2-a^2)^{\frac{1}{2}}},$$

$$\int_a^t J_0(t-y)\frac{J_1((y^2-a^2)^{\frac{1}{2}})}{(y^2-a^2)^{\frac{1}{2}}}dy=a^{-1}[J_0(t-a)-J_0((t^2-a^2)^{\frac{1}{2}})].$$

In part II [Miscellaneous theorems of general kind], not connected with the preceding part, numerous hypergeo-

metric series are summed, e.g.,

$$\sum_{k=0}^{[n/2]} \frac{(-1)^k 2^{n-2k}(n-m-k-1)!}{(n-2k)!k!} = \frac{(-1)^n 2^{2m+1}\Gamma(m+\frac{1}{2})(n-2m-1)!}{\pi^{\frac{1}{2}} n!},$$

$m \leqq \frac{1}{2}(n-1)$. Furthermore, many considerations connected with the iterated operation $x^{-1}d/dx$, e.g.,

$$\left(\frac{1}{x}\frac{d}{dx}\right)^n f(x) = \frac{1}{x^{2n}} \sum_{k=1}^{n} \frac{(-1)^{n-k}(2n-k-1)!}{2^{n-k}(n-k)!(k-1)!} x^k \frac{d^k}{dx^k} f(x), \quad n \geqq 1,$$

and its generalisations are given. [Cf. H. Hadwiger, Comment. Math. Helv. **15**, 353–357 (1943); these Rev. **6**, 65; O. Schlömilch, Compendium der höheren Analysis, v. 2, 2d ed., 1866, p. 9; J. Reine Angew. Math. **32**, 1–7 (1846).]

S. C. van Veen (Delft).

Blanuša, Danilo. A class of integral theorems for Bessel functions. Rad Hrvatske Akademije Znanosti i Umjetnosti. Razred Mat.-Prirodoslov. **86**, 75–152 (1945). (Croatian)

A summary is reviewed above.

de Groot, Sybren R. Un théorème sur les fonctions harmoniques. C. R. Acad. Sci. Paris **224**, 1689–1690 (1947).

The theorem is that the harmonic functions

$$Y_{\alpha,\beta,\gamma}^{(k)} = \frac{(-1)^k}{k!} r^{k+1} \frac{\partial^k}{\partial x^\alpha \partial y^\beta \partial z^\gamma} \frac{1}{r},$$

where α, β, γ are integers, $k=\alpha+\beta+\gamma$, and x, y, z are the coordinates of a radius vector r, satisfy the relation

$$\sum_{\alpha',\beta',\gamma'} \frac{n!}{\alpha'!\beta'!\gamma'!} Y_{\alpha+\alpha',\beta+\beta',\gamma+\gamma'}^{(k+n)} Y_{\alpha',\beta',\gamma'}^{(n)} = \frac{2n!}{2^n(n!)^2} Y_{\alpha,\beta,\gamma}^{(k)},$$

where $n=\alpha'+\beta'+\gamma'$.

M. C. Gray (New York, N. Y.).

Harmonic Functions, Potential Theory

Brelot, Marcel. Le problème de Dirichlet "ramifié." Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **22**, 167–200 (1946).

The author gives a detailed study of topics which he introduced in outline in earlier papers [C. R. Acad. Sci. Paris **221**, 654–656 (1945); **222**, 851–852 (1946); these Rev. **7**, 204, 447]. Earlier studies of the Dirichlet problem for domains with multiple boundary points have been concerned primarily with cases in which the given boundary values have been subjected to restrictions of continuity with respect to the boundary elements. In the present paper the author develops a theory for domains with multiple boundary points corresponding to his treatment of the problem for domains with simple boundary points and discontinuous boundary values [Acta Litt. Sci. Szeged **9**, 133–153 (1939); Bull. Sci. Math. (2) **65**, 72–98 (1941); Ann. Sci. École Norm. Sup. (3) **61**, 301–332 (1944); these Rev. **1**, 121; **7**, 15, 204].

The problem is formulated for a domain Ω with nonpolar complement in the compact space $\bar{R}_\tau$ of $\tau \geqq 2$ dimensions. The author distinguishes between different modes of approach to accessible boundary points by introducing the idea of classes $\mathfrak{L}$ of equivalent curves of approach and describes a metric space whose elements comprise these classes $\mathfrak{L}$ and the interior points of Ω. The Dirichlet problem is then studied for boundary values $f(\mathfrak{L})$ not necessarily continuous. The functions $\underline{H}_f$, $\bar{H}_f$ and H_f are generalized and an integral representation of the solution of the new form of the Dirichlet problem is obtained by the use of the Daniell integral. The paper also includes a discussion of the relation of the new boundary value problem in $\bar{R}_2$ to some topics in the theory of conformal mapping.

F. W. Perkins (Hanover, N. H.).

Brelot, Marcel. Étude générale des fonctions harmoniques ou surharmoniques positives au voisinage d'un point-frontière irrégulier. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **22**, 205–219 (1946).

Bouligand has studied functions harmonic in a domain and zero on the boundary except possibly at one point O. [Fonctions harmoniques, principes de Picard et de Dirichlet, Mémor. Sci. Math., no. 11, Paris, 1926.] In the present paper the author studies related problems in which O is an irregular boundary point of a domain Ω in the compact space $\bar{R}_\tau$ of $\tau \geqq 2$ dimensions. If O is distinct from $\mathfrak{R}_\tau$ (the point at infinity) then any function u subharmonic in Ω has a finite or infinite pseudo-limit at O; if, further, $u(M)/h(\overline{OM})$ has an upper bound in a pseudo-neighborhood of O then u/h has a finite pseudo-limit at O. Here $h(\overline{OM}) = (\overline{OM})^{-(\tau-2)}$ if $\tau > 2$, or $-\log(\overline{OM})$ if $\tau = 2$. Many further theorems are given, including results for the case in which $O = \mathfrak{R}_\tau$ and for a general form of Green's function.

F. W. Perkins.

Cartan, Henri. Théorie générale du balayage en potentiel newtonien. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **22**, 221–280 (1946).

The author uses some of his earlier results and techniques [Bull. Sci. Math. (2) **66**, 126–132, 136–144 (1942); Bull. Soc. Math. France **69**, 71–96 (1941); **73**, 74–106 (1945); these Rev. **6**, 86; **7**, 447, 620] to obtain general "sweeping-out" processes for distributions of (positive) mass on arbitrary sets A in n-space. Although the author restricts the discussion to the "Newtonian case," the results can be extended, as he points out, to "potentials of order α." Let x be a point in R^n, $n \geqq 2$; here R^n denotes Euclidean n-space, $n \geqq 3$, while R^2 denotes the unit disc $|x| < 1$. Let $\mathfrak{C}$ denote the class of finite nonnegative continuous functions $f(x)$, each of which vanishes outside some (not necessarily the same) compact set, and let $\mathfrak{s}$ denote the class of functions $g(x)$ each of which is the limit of a monotone increasing sequence of functions of $\mathfrak{C}$. A finite functional $\mu(f)$, defined on $\mathfrak{C}$, will be called a distribution of (positive) mass if and only if (i) $\mu(f) \geqq 0$, $f \in \mathfrak{C}$, and (ii) $\mu(f_1+f_2) = \mu(f_1)+\mu(f_2)$, $f_1, f_2 \in \mathfrak{C}$. If $g(x) \in \mathfrak{s}$, and if μ is a distribution of mass, then the (not necessarily finite) integral $\int g d\mu$ is defined to be

$$\int g(x)d\mu(x) = \text{l.u.b.}_f [\mu(f) \mid f \in \mathfrak{C},\ f(x) \leqq g(x)].$$

In particular, the potential of a distribution μ is defined to be the function $U^\mu(x) = \int \varphi(x,y)d\mu(y)$, where $\varphi(x,y) = |x-y|^{2-n}$ if $x, y \in R^n$, $n \geqq 3$; $\varphi(x,y) = \log|(1-x\bar{y})/(x-y)|$ if $x, y \in R^2$. The energy of μ is defined to be the integral $\int U^\mu d\mu$.

"Sweeping-out" processes are defined in two steps; first for the distributions μ having finite energy, then for a larger class. Let $\mathcal{E}$ be the set of μ for which $\int U^\mu d\mu < \infty$ and let $\mathcal{E}^*$ be the set of differences $\alpha = \mu - \nu$, $\mu \in \mathcal{E}$, $\nu \in \mathcal{E}$. If $\alpha = \mu - \nu$ and $\alpha' = \mu' - \nu'$, in $\mathcal{E}^*$, are identified whenever $\mu+\nu' = \mu'+\nu$, then $\mathcal{E}^*$ has the structure of a vector space over the real numbers;

moreover, $(\alpha, \alpha') = \int (U^p - U^r)(d\mu' - d\nu')$ defines a scalar product in $\mathcal{E}^*$. If the norm $\|\alpha\| = (\alpha, \alpha)^{\frac{1}{2}}$ is introduced in $\mathcal{E}^*$, then $\mathcal{E}^*$ becomes a "pre-Hilbert" space; that is, $\mathcal{E}^*$ is not complete but its completion is a (real) Hilbert space. Now let K be a compact set in R^n and consider the set $\mathcal{E}_K$ of distributions having finite energy carried by K. In the topology of $\mathcal{E}^*$, $\mathcal{E}$ is a complete convex subset of $\mathcal{E}^*$ while $\mathcal{E}_K$ is a closed convex subset of $\mathcal{E}$. Hence, for a fixed $\mu \in \mathcal{E}$, there exists a unique element $\mu_K \in \mathcal{E}_K$ that minimizes $\|\mu - \nu\|$ among the elements $\nu \in \mathcal{E}_K$; the element μ_K is the "projection" of μ on $\mathcal{E}_K$. The distribution μ_K is defined to be that obtained by "sweeping-out" μ on K. Now let A be an arbitrary set in R^n and let $\mathcal{E}'_A = \cup_K \mathcal{E}_K$, where K is compact and $K \subset A$, that is, $\mathcal{E}'_A$ is the closure of that subset of $\mathcal{E}$ consisting of those distributions of finite energy carried by A. Then $\mathcal{E}'_A$ is a closed convex subset of $\mathcal{E}$ and hence the projection μ'_A of μ on $\mathcal{E}'_A$ exists; μ'_A is that distribution obtained by the "inner sweeping-out" of μ on A. If we define $\mathcal{E}^e_A = \cap_B \mathcal{E}'_B$, where B is open and $B \supset A$, then $\mathcal{E}^e_A$ is a closed convex subset of $\mathcal{E}$ and hence the projection μ^e_A of μ on $\mathcal{E}^e_A$ exists; μ^e_A is the distribution obtained by the "outer sweeping-out" of μ on A. One should note the following characterizations of μ'_A and μ^e_A, respectively: (1) $(\mu'_A - \mu, \lambda) = 0$ for each $\lambda \in \mathcal{E}'_A$, (2) $(\mu^e_A - \mu, \lambda) = 0$ for each $\lambda \in \mathcal{E}^e_A$. Conditions (1) and (2) state that μ'_A and μ^e_A are the "feet" of the perpendiculars drawn from μ to $\mathcal{E}'_A$ and $\mathcal{E}^e_A$, respectively. From (1) and (2) and the "reciprocity law," one obtains $(\lambda, \mu'_A) = (\mu, \lambda'_A)$ and $(\lambda, \mu^e_A) = (\mu, \lambda^e_A)$, for all $\lambda, \mu \in \mathcal{E}$. Now let $\mathfrak{M} = [\mu \mid U^\mu \neq \infty]$. Then for each $\mu \in \mathfrak{M}$ there exists a unique $\nu_1 \in \mathfrak{M}$ such that $(\nu_1, \lambda) = (\mu, \lambda'_A)$, for each $\lambda \in \mathcal{E}$, and there exists a unique $\nu_2 \in \mathfrak{M}$ such that $(\nu_2, \lambda) = (\mu, \lambda^e_A)$, for each $\lambda \in \mathcal{E}$. The distribution ν_1 (ν_2) is denoted by μ'_A (μ^e_A) and is said to be obtained by the "inner (outer) sweeping-out" of μ on A.

One can now define capacities and capacitary distributions in the usual way. For example, let $\mu \in \mathcal{E}$ and let μ satisfy the condition (*) $U^\mu = 1$, except possibly on a set which is of ν-measure zero for each distribution $\nu \in \mathcal{E}'_A$. Then the "inner sweeping-out" of μ on A yields the unique distribution μ'_A. The distribution μ'_A is independent of the particular μ chosen, $\mu \in \mathcal{E}$, so long as it satisfies (*); this distribution is denoted by γ'_A and called the inner capacitary distribution of A; $\|\gamma'_A\|^2 = \int d\gamma'_A$ is called the inner capacity of A. A similar development can be given for the outer capacitary distribution γ^e_A and the outer capacity $\|\gamma^e_A\|^2$. Of course, in order for either γ'_A or γ^e_A to exist there must be a μ in $\mathfrak{M}$ satisfying (*); γ'_A and γ^e_A exist for bounded sets, for example, as well as for open sets and for closed sets. Indeed, for open sets and for closed sets, $\gamma'_A = \gamma^e_A$. To define regular points, one starts with a distribution ϵ_x consisting of a unit mass placed at the point x; x is said to be an inner regular point for a set A if and only if $(\epsilon_x)'_A = \epsilon_x$; and x is said to be an outer regular point for A if and only if $(\epsilon_x)^e_A = \epsilon_x$.

With the preceding basic notions in hand, the author proceeds to develop them along modern potential-theoretic lines, using modern Hilbert space techniques. His approach to the problems of modern potential theory consists in approaching them from the standpoint of the mass distributions rather than from the standpoint of the potentials; this, with the aid of various topologies on the spaces $\mathcal{E}^*$ and $\mathfrak{M}$, allows him to obtain results which show the natural connections between the basic ideas in potential theory, without the aid of superfluous hypotheses. Recent results due to Brelot, de la Vallée Poussin, Frostman and others are shown to be included in the author's more general and more easily derived results. *M. Reade.*

Nicolesco, Miron. Nouvelles recherches sur les fonctions polyharmoniques. Disquisit. Math. Phys. 1, 43–56, 173–187 (1940).

Let D be a domain in n-space. Then $V(M) \equiv V(x_1, \cdots, x_n)$ is said to be p.o. p (polyharmonic of order p) in D if and only if $\Delta^p V = 0$ in D; here Δ denotes the Laplacian. It is known that if $V(M)$ is p.o. p in D, and if M_0 is a fixed point in D, then there exists a unique set of functions $\{U_k(M)\}$, harmonic and regular in D, such that

$$(1) \qquad V(M) \equiv U_1(M) + r^2 U_2(M) + \cdots + r^{2p-2} U_p(M);$$

here $r = \overline{M_0 M}$ [Nicolesco, Bull. Math. Soc. Roumaine Sci. 37, 83–87 (1935)]. The expansion (1) is called the Almansi expansion of $V(M)$ about M_0. Since a harmonic function $U(M)$ can be expanded about an isolated singularity M_0 in a series of spherical harmonics,

$$(2) \qquad U(M) = c_0 + d_0 r^{-n+2} + \sum_{\nu=1}^{\infty} (d_\nu r^{-n+2-\nu} + c_\nu r^\nu) S_\nu(m),$$

where m is the point at which the ray r meets the unit sphere about M_0, the question arises as to the possibility of an Almansi expansion (1) for $V(M)$ p.o. p about an isolated singularity M_0, where the $\{U_k(M)\}$ have the form (2). This and the following two papers are devoted to that problem. If we say that $V(M)$, p.o. p, is A-decomposable about an isolated singularity M_0 whenever $V(M)$ admits an expansion (1), where the $\{U_k(M)\}$ have the form (2) about M_0, then the author's most general result may be stated as follows. Let $V(M)$ be p.o. p in D, except for an isolated singularity at M_0. If n is odd, then $V(M)$ is A-decomposable about M_0; if n is even, $p < n/2$, then $V(M)$ is A-decomposable about M_0. Moreover, if n is even, $p \geqq n/2$, then $V(M)$ is the sum of two functions which are p.o. p in D, such that one function is A-decomposable about M_0, while the other has a logarithmic singularity at M_0. The author's methods depend upon the usual series expansions for harmonic functions in n variables. *M. Reade* (Ann Arbor, Mich.).

Nicolesco, Miron. Remarques sur mon mémoire: Recherches sur les fonctions polyharmoniques. Disquisit. Math. Phys. 1, 189–190 (1940).

In this note the author completes his proof of the following theorem. If $V(M)$ is p.o. p [see the preceding review] in D, $n \geqq 4$, and if $V(M)$ is regular in D, then the harmonic components in the Almansi expansion of $V(M)$ about a point M_0 in D are regular at M_0. The author credits Picone for calling his attention to the omission of "$n \geqq 4$" from the hypothesis of the theorem; indeed, in a letter to the author, Picone gave examples to show that the corresponding theorems do not hold for $n = 2, 3$. *M. Reade.*

Nicolesco, Miron. Le problème de Lauricella pour les domaines hypersphériques. Disquisit. Math. Phys. 1, 321–338 (1941).

The author considers the following general problem. Given a domain D (in n-space), with boundary F, and a set of functions $f_0(P), f_1(P), \cdots, f_{p-1}(P)$, continuous for $P \in F$, to determine $V(M)$ p.o. p in D [see the preceding review], such that $\lim_{M \to P} (-1)^k \partial^k V / \partial r^k = f_k(P)$, $k = 0, 1, \cdots, p-1$. Lauricella proposed and solved the problem for $n = 2$ [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 18_2, 5–15 (1909)]. The author gives a solution for (hyper)spheres, for general n; his solution is given in finite form, in contrast to Picone's solution in series form [Ann. Scuola Norm. Super. Pisa (2) 5, 213–288 (1936)]. In his construction of the

solution, the author makes extensive use of Almansi's expansion for a function p.o. p in a domain and of the Poisson representation for a function harmonic within a (hyper)-sphere. The results are too lengthy to quote in detail.

M. Reade (Ann Arbor, Mich.).

Friedrichs, K. O. A theorem of Lichtenstein. Duke Math. J. 14, 67–82 (1947).

L'auteur généralise un théorème de Lichtenstein [J. Reine Angew. Math. 141, 12–42 (1912)] concernant le potentiel logarithmique d'une couche dont la densité est de carré sommable: il y a presque partout des dérivées secondes et un laplacien égal à la densité. L'auteur donne des résultats analogues pour le potentiel newtonien à n variables en montrant que s'il y a un laplacien en un certain sens généralisé, il y a des dérivées secondes en un sens convenablement étendu. *M. Brelot* (Grenoble).

Differential Equations

Pompeiu, D. Sur l'équation différentielle linéaire du premier ordre. Bull. Sci. École Polytech. Timişoara 10, 203–204 (1941).

Halpern, S. Sur les asymptotes des solutions de l'équation $y'=f(x, y)$. C. R. (Doklady) Acad. Sci. URSS (N.S.) 54, 383–386 (1946).

The author demonstrates the following theorem. Suppose that in the semi-strip $x\geqq\alpha$, $\beta-m\leqq y\leqq\beta+m$ the following conditions are satisfied: (1) $f(x, y)$ is continuous; (2) the equation $f(x, y)=0$ determines a continuous curve $y=b(x)$ which for $x\geqq\alpha$ lies in the strip and has the line $y=\beta$ as an asymptote; (3) there exists a function $x=\omega(y)$, continuous and monotonic in the intervals $\beta-m<y<\beta$ and $\beta<y<\beta+m$ such that curves representing $x=\omega(y)$ lie in the semi-strip and such that $\omega(\beta-0)=\omega(\beta+0)=+\infty$; moreover, for each fixed value of y, $\inf_{x>\omega(y)}|f(x, y)|\geqq K(y)\geqq 0$, where $1/K(y)$ is an integrable function in any segment not containing the point β. If $f(x, y)$ is negative above $y=b(x)$ and positive below then each solution of $y'=f(x, y)$ having one point in the semi-strip has $y=\beta$ as asymptote. The author also considers the situation when $f(x, y)$ has opposite signs from those indicated. *N. Levinson* (Cambridge, Mass.).

***Minorsky, N. Introduction to Non-Linear Mechanics. Topological Methods. Analytical Methods. Non-Linear Resonance. Relaxation Oscillations.** J. W. Edwards, Ann Arbor, Mich., 1947. xiv+464 pp. $5.00.

This book in four parts as indicated in the title consists of reports originally published by the David Taylor Model Basin. The first three parts have already been reviewed [these Rev. 8, 207]; hence only part IV is reviewed here. Part IV consists of 67 pages. "The text of part IV follows the presentation in 'Theory of Oscillations' by Andronow and Chaikin, Moscow, 1937."

The first three chapters of part IV take up the discontinuous theory of relaxation oscillations with applications to electrical systems with one and several degrees of freedom. A short chapter on mechanical relaxation oscillations follows. The next chapter is on oscillations maintained by periodic impulses with the clock as a mechanical example and a triode oscillator with quasi-discontinuous grid control as an electrical example. The closing chapter takes up the effect of parasitic parameters with particular reference to stability.

N. Levinson (Cambridge, Mass.).

de Bruijn, N. G. A note on van der Pol's equation. Philips Research Rep. 1, 401–406 (1946).

The author proves that the periodic solutions of the van der Pol equation $u''+\epsilon(u^2-1)u'+u=0$ are given by the familiar series $u(t, \epsilon)=\sum\epsilon^n u_n(\omega t)$, where the u_n are periodic of period 2π and $\omega=\sum\epsilon^n b_n$. He shows in fact that for small ϵ there exists an infinite strip in the complex t plane in which $u(t, \epsilon)$ can be analytically continued. He then shows by direct computation that Shohat's results for $\epsilon\gg 1$ [J. Appl. Phys. 15, 568–574 (1944); these Rev. 6, 1], which Shohat computed using four terms of a series, cease to correspond well with known results if six terms of the series are used.

N. Levinson (Cambridge, Mass.).

Almeida Meneses, Pablo Rogerio. Subresonance for an equation of second order with forced and periodic vibrations, when the friction and the elastic force are not linear in the velocity. Revista Ci., Lima 49, 71–80 (1947). (Spanish)

This note consists of the opening pages of a thesis on the subject announced in the title and contains nothing beyond some familiar introductory material. The publication of the thesis will be continued. *L. A. MacColl.*

Crandall, Stephen H. On restoring forces which admit forcing terms of non-critical amplitude. Bull. Amer. Math. Soc. 53, 633–636 (1947).

A forcing term $f(t)$ of exact period T is called of non-critical amplitude with respect to a restoring force $g(x)$ if the conservative system $\ddot{x}+g(x)=f(t)$ admits a periodic solution $x(t)$ having the following property: to any k of a continuous range, there exists a constant h, such that $y=h\cdot x(t)$ is a solution of $\ddot{y}+g(y)=k\cdot f(t)$. The author assumes that $g(x)$ is odd and possesses a derivative. He proves that for the existence of a forcing term of noncritical amplitude it is necessary that the restoring force $g(x)$ is of the form $ax+bx^n$ or $ax+bx\log x$. Unless $b=0$ the forcing term $f(t)$ is essentially uniquely determined. The results are applied to Duffin's equation $\ddot{x}\pm x+bx^3=f(t)$. The proofs given are not absolutely complete: the ranges over which $x(t)$ and k vary are not discussed, occurrences of $h=k$ for certain k are not taken into account. Finally, it should be remarked that the condition that $g(x)$ admits a derivative is too restrictive; integrability is already sufficient.

F. Bohnenblust (Pasadena, Calif.).

Germay, R.-H.-J. Sur les systèmes complètement intégrables d'équations aux différentielles totales à coefficients linéaires. Bull. Soc. Roy. Sci. Liége 15, 284–292 (1946).

A method for solving the completely integrable system of total differential equations of the form

$$dz_j=\sum_{k=1}^{n}\left\{\sum_{\alpha=1}^{p}A_{j\alpha}^{(\alpha)}(x_1, \cdots, x_n)z_\alpha+A_{j\alpha}^{(0)}(x_1, \cdots, x_n)\right\}dx_k, \qquad j=1, \cdots, p,$$

$$z_j(\overset{0}{x}_1, \cdots, \overset{0}{x}_n)=\overset{0}{z}_j,$$

is given using functions associated with the functions of Riemann. The present treatment is somewhat different from the previous methods given by the author [Bull. Sci. Math. (2) 63, 272–278 (1939); Bull. Soc. Roy. Sci. Liége 11, 151–163 (1942); these Rev. 1, 119; 7, 120]. *F. G. Dressel.*

Germay, R.-H.-J. Extension aux systèmes linéaires d'équations aux différentielles totales complètement intégrables, de propriétés fondamentales des systèmes différentiels linéaires. Bull. Soc. Roy. Sci. Liége **15**, 348–357 (1946).

Several well-known theorems for systems of ordinary linear differential equations are extended to linear total differential equations which are completely integrable. The following statement will serve to illustrate the types of theorems. If the system

$$(*) \qquad du_i=\sum_{j=1}^{2}\{A_{ij}^{(1)}(x_1, x_2)u_1+A_{ij}^{(2)}(x_1, x_2)u_2\}dx_j, \quad i=1, 2,$$

is completely integrable, and u_{11}, u_{12} and u_{21}, u_{22} are two systems of solutions of (*) for which the determinant $\Delta(x_1, x_2)=|u_{ij}(x_1, x_2)|$ does not vanish at (x_1^0, x_2^0), then

$$\Delta(x_1, x_2)=\Delta(x_1^0, x_2^0)\exp\left\{\sum_{i=1}^{2}\int_{x_i^0}^{x_i}(A_{1i}^{(1)}+A_{2i}^{(2)})dx_i\right\}.$$

F. G. Dressel (Durham, N. C.).

Germay, R. H. J. Sur l'élimination des paramètres dans la méthode des caractéristiques de Cauchy pour l'intégration des équations aux dérivées partielles du premier ordre. Ann. Soc. Sci. Bruxelles. Sér. I. **61**, 99–105 (1947).

Under sufficient conditions the first order partial differential equation

$$(1) \qquad F(x_1, \cdots, x_n; z; p_1, \cdots, p_n)=0$$

may be solved for p_1:

$$(2) \qquad p_1=f(x_1, \cdots, x_n; z; p_2, \cdots, p_n).$$

The solution of equation (2) can be obtained by using its associated characteristic differential equations. The function f, in general, cannot be given in finite form. This paper gives two methods for solving (1) that do not require the knowledge of the function f. *F. G. Dressel.*

Vessiot, Ernest. Sur une théorie générale de la réductibilité des équations et systèmes d'équations finies ou différentielles. Ann. Sci. École Norm. Sup. (3) **63**, 1–22 (1946).

The first part of this paper describes a treatment of a general class of families of systems of equations. The object of the second part is to furnish illustrations of the treatment from familiar branches of the theory of differential equations. This generalizes previous work of the author's on algebraic equations [C. R. Acad. Sci. Paris **210**, 159–161 (1940); these Rev. **1**, 194]. The families of systems studied are those for which each system has a complete solution, that is, a solution from which the particular solutions can be obtained by specialization and by rational operations, including elimination and differentiation. It is further required that the systems of such a family admit auxiliary transformations into one another. The complete solutions are considered as particular solutions of an associated system; a system is reducible relative to a domain of rationality if its associated system has a subsystem rational in the domain. In each family an arbitrary system is selected and the connection between its associated system and subsystems and those of other members of the family is discussed by means of certain of the auxiliary transformations, furnishing results analogous to those of the Galois theory.

The illustrations are (a) systems of n first-order differential equations in n unknowns of the form $x_\alpha'=\theta_\alpha(t, x_1, \cdots, x_n)$, $\alpha=1, \cdots, n$; (b) systems composed of one nth order differential equation in one unknown of the form

$$x^{(n)}=\theta(t, x, x', \cdots, x^{(n-1)});$$

(c) partial differential equations of the form

$$\partial z/\partial t=\theta(t, x_1, \cdots, x_n; z, \partial z/\partial x_1, \cdots, \partial z/\partial x_n).$$

The discussion is somewhat sketchy; the author remarks: "Nous excluons dans cette note, tous les cas d'exception."

H. Levi (New York, N. Y.).

Finzi, A. Sur les systèmes d'équations aux dérivées partielles, qui, comme les systèmes normaux, comportent autant d'équations que de fonctions inconnues. III, IV. Nederl. Akad. Wetensch., Proc. **50**, 288–297, 351–356 = Indagationes Math. **9**, 178–187, 209–214 (1947).

[For parts I and II see the same Proc. **50**, 136–142, 143–150 = Indagationes Math. **9**, 99–105, 106–113 (1947); these Rev. **8**, 466.] Un système anormal de n équations d'ordre h à n fonctions inconnues φ est caractérisé, on l'a vu, par le fait qu'une certaine combinaison des dérivées d'ordre $h(n-1)$ des équations $F=0$ données est d'ordre inférieur à hn par rapport aux φ; si en joignant cette équation à $n-1$ des $F=0$ (dérivées $hn-h-1$ fois) on obtient un système, d'ordre $h^{(1)}$, encore anormal, et si en recommençant cette opération, les systèmes d'ordres $h^{(2)}, h^{(3)}, \cdots, h^{(l)}$ sont anormaux, l étant supérieur à nh, l'auteur montre, par un dénombrement, qu'il existe entre les F et leurs dérivées une relation identique. A moins d'incompatibilité du système proposé, il montre ensuite qu'il est loisible d'imposer à sa solution une condition différentielle supplémentaire $Z=0$ dans tout le domaine de définition (condition soumise éventuellement à certaines restrictions d'une caractère d'ailleurs très large).

Revenant ensuite au cas où l'opération qui a été rappelée conduit à un système normal, l'auteur complète le théorème d'existence qu'il avait précédemment donné en montrant que sous certaines conditions, très générales, il suffit, au lieu de se donner sur $x^0=a$ les φ et leurs $h^{(1)}-1$ premières dérivées assujetties à $(h^{(1)}-h)n+l$ conditions, de se donner les φ et leurs $h-1$ premières dérivées assujetties à l conditions. Il utilise comme exemple le système de l'applicabilité des surfaces; il traite ce système comme l'avait traité Janet [Ann. Soc. Polon. Math. **5**, 38–43 (1927)], évidemment sans avoir eu connaissance de ce travail.

L'auteur étend ensuite ses résultats au cas où le déterminant fondamental d'ordre n, qui intervient avant tout dans l'étude du système $F=0$ proposé, se trouve être de rang non plus $n-1$, mais $n-t$ quelconque.

Le système des 10 équations du 2e ordre à 10 inconnues que constituent les équations de la gravitation d'Einstein lui donne une application intéressante des résultats précédents. Deux cas se présentent où interviennent respectivement les deux types fondamentaux. Dans le second, le rang du déterminant fondamental étant 6, il y a 4 relations identiques entre les premiers membres des équations d'Einstein; les quatre conditions nécessaires qui en résultent sont d'ailleurs suffisantes. Et, naturellement, la solution du système n'est plus univoquement determinée par la connaissance des valeurs initiales. *M. Janet* (Paris).

Duffin, R. J. Lower bounds for eigenvalues. Physical Rev. (2) **71**, 827–828 (1947).

Referring to the n-dimensional Schroedinger equation

$$\Delta\psi+(\lambda-V)\psi=0$$

over a region R, with $\psi=0$ on the boundary, it is shown by

some simple relations that $\lambda \geqq \min [V-u^{-1}\Delta u]$ in R, where u is any nonnegative function having bounded second derivatives in R. *R. E. Langer* (Madison, Wis.).

Wassermann, G. D. On the theory of boundary perturbations. Philos. Mag. (7) **37**, 563–571 (1946).

When the solution of a differential equation satisfying given boundary conditions is known, we may attempt the solution of a problem derived from the one already solved by slightly changing (perturbing) the differential equation, the boundary conditions and the surface where these conditions hold. The problems of this kind attacked here are as follows. Given the eigenvalue $K_{n,0}$ and the eigenfunction $\psi_{n,0}$, satisfying the equation $H_0\psi_{n,0}+K_{n,0}\psi_{n,0}=0$ (where $H_0=\nabla^2+f(x)$, $f(x)$ representing a function of the coordinates x_1, x_2, x_3) and the condition $\partial\psi_{n,0}/\partial n=0$ on the given surface S_0, to determine the solution ψ of $H_0\psi+K_n\psi=0$, satisfying on a slightly modified surface S the condition $\partial\psi/\partial n=F(x)\psi$, $|F(x)|$ being small compared to unity. Given the eigenvalue and eigenfunction, satisfying the same equation and the condition $\psi_{n,0}=0$ on S_0, to determine the solution satisfying the condition $\psi=F(x)\partial\psi/\partial n$ on S.

The solution is sought in the form of a series by introducing a displacement operator R transforming the point x_i on the surface S_0 into $x_i+X(x_i)$ on S, so that $Rf(x)=f(x+X)=(1+\mu^{\mathrm{I}}+\mu^{\mathrm{II}}+\cdots+\mu^r+\cdots)f(x)$; ψ_n and K_n are expanded in the forms $\psi_n=\psi_{n,0}+\psi_{n,1}+\cdots+\psi_{n,r}+\cdots$, $K_n=K_{n,0}+K_{n,1}+\cdots+K_{n,r}+\cdots$. Furthermore, the differential equation also is perturbed, into $H\psi_n+K_n\psi_n=0$, where $H=H_0+H_1+\cdots+H_r+\cdots$.

The results obtained are substantially in agreement with those obtained by Feshbach [Physical Rev. (2) **65**, 307–318 (1944); these Rev. **5**, 266] by means of an integral equation involving a Green's function. Questions of convergence are mentioned but not discussed. *H. Bremekamp* (Delft).

Schaaf, S. A. On the superposition of a heat source and contact resistance. Quart. Appl. Math. **5**, 107–111 (1947).

Let there be a source of heat as well as a thermal contact resistance at an interface between two solids, as in the case of friction between moving bodies at their surface of contact. The author develops a simple boundary condition for the temperature function at such a surface, one which seems to describe fairly well the conditions under either lubricated or dry friction. To illustrate the use of the condition he determines the temperature distribution in two semi-infinite solids of different materials, having a heat source and a resistance at their common face. *R. V. Churchill.*

Griffith, Miriam V., and Horton, G. K. The transient flow of heat through a two-layer wall. Proc. Phys. Soc. **58**, 481–487 (1946).

The boundary value problem for the temperature function $U(x, t)$ in a composite wall consisting of a layer of finite thickness in contact with a semi-infinite layer, when there is a constant flux of heat through the outer surface, is solved here by using Laplace transforms. The initial temperature is assumed uniform and the two layers are assumed to be in perfect thermal contact. The function $U(x, t)$ is determined as a series of error functions. The authors present a graphical analysis of the control of the surface temperature by varying the thickness and the thermal properties of the layer of finite thickness. They are interested in this analysis in connection with the rapid heating of rooms by a radiating source of heat within the room. *R. V. Churchill.*

Paterson, S. The heating or cooling of a solid sphere in a well-stirred fluid. Proc. Phys. Soc. **59**, 50–58 (1947).

A solid sphere initially at a uniform temperature is immersed in a prescribed mass of fluid initially at a different uniform temperature. The fluid is then kept stirred to such an extent that its temperature remains uniform and equal to the surface temperature of the sphere. The boundary value problem for the temperature in the sphere, a function of time and distance from the center, is set up in terms of dimensionless quantities and solved by operational methods. Infinite series and asymptotic forms of the solution are developed in order that actual computations of temperatures can be carried out for all values of the variables and parameters. Tables of roots of the mixed equation which arises here, and tables and graphs of the temperatures, are presented; also the sums of several special infinite series of functions are noted. *R. V. Churchill* (Ann Arbor, Mich.).

Vernotte, Pierre. Extension de la méthode de Fourier à l'étude des systèmes complexes dans lesquels certains milieux sont séparés par une résistance de passage. C. R. Acad. Sci. Paris **224**, 1416–1418 (1947).

An identity involving volume and surface integrals is established, with the aid of the symmetric form of Green's theorem, for the function $\phi(x, y, z)$ where $\phi(x, y, z)\epsilon^{-mt}$ is a solution of the heat equation. The general form of the identity contains a term corresponding to an interface resistance to the flow of heat and a term corresponding to a linear transfer of heat at the outer surface. The complete temperature problem is not stated. *R. V. Churchill.*

Brown, R. L. A problem in non-steady heat conduction. Philos. Mag. (7) **37**, 318–322 (1946).

A brief study is made of the transient thermal insulation furnished by a wall of finite thickness made of homogeneous conducting material. The temperature of one face is kept fixed, equal to the uniform initial temperature of the wall, and the flux through the other face is uniform. The solution of the boundary value problem, carried out here by the method of sources and images, is well known and easily obtained by operational methods [note, for example, problems 11 and 12 on p. 198 of the reviewer's textbook Modern Operational Mathematics in Engineering, McGraw-Hill, New York, 1944; these Rev. **5**, 267]. *R. V. Churchill.*

Datzeff, Assène. Sur le refroidissement d'un corps non homogène. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 111–114 (1947).

The temperature distribution is determined in a composite solid consisting of two semi-infinite bodies whose faces are in perfect thermal contact. The initial temperature is an arbitrary function of the distance from the interface. This problem was previously solved by W. A. Mersman [Bull. Amer. Math. Soc. **47**, 956–964 (1941); these Rev. **3**, 128]. *R. V. Churchill* (Ann Arbor, Mich.).

Masket, A. Victor. Forced vibrations of a whirling wire. Philos. Mag. (7) **37**, 426–432 (1946).

The author [J. Acoust. Soc. Amer. **18**, 216 (1946); these Rev. **8**, 80] gave a brief account of the Laplace transform method of solving

$$\frac{\partial}{\partial y}\left\{(1-y^2)\frac{\partial v}{\partial y}\right\}=\frac{\partial^2 v}{\partial \tau^2}$$

under the following conditions: when $\tau=0$, $v=0$ and

$\partial v/\partial\tau=0$ for $0\leqq y<\alpha l<1$; when $y=0$, $v=0$ for all τ; when $y=\alpha l$, $v=B\sin\beta\tau$. The present paper gives a more detailed account. *E. T. Copson* (Dundee).

Integral Equations

Zaanen, A. C. On the theory of linear integral equations. VII. Nederl. Akad. Wetensch., Proc. **50**, 357–368 = Indagationes Math. **9**, 215–226 (1947).

The author extends the results of paper II of the series [same Proc. **49**, 205–212 = Indagationes Math. **8**, 102–109 (1946); these Rev. **8**, 28] to systems of linear integral equations. No new questions of principle are involved.

F. Smithies (Cambridge, England).

* **Mushelišvili, N. I. Singulyarnye Integral'nye Uravneniya. Graničnye Zadači Teorii Funkcil i Nekotorye ih Priloženiya k Matematičeskoĭ Fizike. [Singular Integral Equations. Boundary Problems of the Theory of Functions and Certain of their Applications to Mathematical Physics].** OGIZ, Moscow-Leningrad, 1946. 448 pp.

The book has its origin in the author's reports at the seminar of the Tbilissi Mathematical Institute; it represents an exposition of the theory of Cauchy integrals as bearing on the theory of singular integral equations involving such integrals, largely developed by the author and his pupils. The following is some of the notation used. The author denotes by L a finite number of bounded open and simple closed curves in the complex plane, without common points; each open arc and closed curve has a continuously turning tangent. A singular operator is defined by

$$(1)\qquad K^*\varphi=A(\tau)\varphi(\tau)+(\pi i)^{-1}\int_L(t-\tau)^{-1}K(\tau,t)\varphi(t)dt,$$

where t, τ are on L; $A(\tau)$, $K(\tau,t)$, $B(\tau)=K(\tau,\tau)$ belong to H (a Hölder class); $A\pm B\neq 0$ on L; (1) is expressed in the form

$$(2)\qquad K^*\varphi=K^{0*}\varphi+(\pi i)^{-1}\int\kappa(\tau,t)\varphi(t)dt,$$

where

$$K^{0*}\varphi=A(\tau)\varphi(\tau)+(\pi i)^{-1}B(\tau)\int(t-\tau)^{-1}\varphi(t)dt,$$

$\kappa(\tau,t)(t-\tau)=K(\tau,t)-K(\tau,\tau)$.

The main problem is the singular integral equation (3) $K^*\varphi=f$, $f\in H$, in the field of H, with suitable conditions of behaviour near the end points of L. The integrations are in the sense of Cauchy principal values. Special cases of (3) became a subject of study soon after the publication of the Fredholm theory of integral equations [H. Poincaré on the theory of the tides, Hilbert on Riemann-Hilbert boundary value problems]. Problems (I) ((3) for closed curves), (II) ((3) for open arcs) are handled in succession; their synthesis is not difficult; the study of each is preceded by solution of appropriate Riemann-Hilbert boundary value problems (with the aid of Plemelj formulas and of a study of the behaviour of integrals $\int(t-\tau)^{-1}\varphi(t)dt$, $\varphi\in H$, near L and near the end point of L).

Methods of regularization [Vecoua, for problem (I)] are widely used; these consist of applying to (3) an operator transforming (3) into a regular Fredholm integral equation. Of importance is also the use of indices, which are certain numbers used in uniformizations [Gahoff employed this device for the first time in a special case]. Using suitable classes of solutions (depending on the values of the coefficients near and at the end points of L) the theories for problems (I), (II) are made definitive. It is indicated how to handle the situation when the coefficients in (3) have a finite number of discontinuities, as well as in the case of a finite number of "corner" points on L. Many applications are given to problems of potential, elasticity, electrostatics, the airplane wing, etc. The book concludes with an extensive bibliography. [The case when the constituent curves of L intersect in a finite number of points has been treated in a work by the reviewer [Trans. Amer. Math. Soc. **60**, 167–214 (1946); these Rev. **8**, 211], which appeared after the publication of the book under review].

W. J. Trjitzinsky (Urbana, Ill.).

Marshak, R. E. The variational method for asymptotic neutron densities. Physical Rev. (2) **71**, 688–693 (1947).

This paper is concerned with a variational method of obtaining approximate solutions of a nonhomogeneous integral equation $q_0(x)=f(x)+\int K(x,y)q_0(y)dy$, where the nucleus $K(x,y)$ is positive and symmetric. It can be shown that the functional

$$I=\frac{\int q(x)\{q(x)-\int K(x,y)q(y)dy\}dx}{\{\int q(x)f(x)dx\}^2}$$

has a stationary value when $q=q_0$, the stationary value being $1/\int q_0 f dx$. Thus if a rough trial solution $q(x)$ involving one or more arbitrary constants is inserted in I, and if the constants are chosen to make I stationary, an approximation to $\int q_0 f dx$ is obtained. In the particular type of problem discussed here, the asymptotic behaviour of $q_0(x)$ can be shown to be such that $q_0(\infty)=k_1\int q_0 f dx+k_2$, where k_1 and k_2 are numerical constants. Hence the method gives an approximation to $q_0(\infty)$.

The first problem discussed is that of Milne's integral equation

$$q_0(x)=\tfrac{1}{2}E_3(x)+\tfrac{1}{2}\int_0^\infty E_1(|x-y|)q_0(y)dy,$$

where $E_n(x)=\int_1^\infty t^{-n}e^{-xt}dt$ [see E. Hopf, Mathematical Problems of Radiative Equilibrium, Cambridge University Press, 1934]. It is shown by a physical argument that

$$q_0(\infty)=\tfrac{3}{8}+\tfrac{3}{2}\int_0^\infty q_0(x)E_3(x)dx$$

and an expression for $q_0(\infty)$, correct to 0.3%, is obtained by using a constant as trial solution in the functional I.

The second application is to obtain an estimate of the asymptotic density of neutrons in an infinite isotropically scattering (but noncapturing) medium supporting a constant flux from infinity and surrounding a perfectly absorbing spherical core. It turns out that, if one takes a constant as trial solution in the functional I, the corresponding results are about as good as the P_3 approximation in the spherical harmonic solution. *E. T. Copson* (Dundee).

Davison, B. A remark on the variational method. Physical Rev. (2) **71**, 694–697 (1947).

The author proves the following theorem [cf. the preceding review]. Let $q_0(x)$ be the bounded solution of the integral equation

$$q_0(x)=f(x)+\lambda\int_0^\infty K(x,y)q_0(y)dy$$

in which λ is a fixed positive constant, $K(x,y)$ is positive and symmetrical, $f(x)$ is bounded and $\int_0^\infty|f(x)|dx$ exists.

Then the functional

$$I=\frac{\int_0^\infty q(x)\{q(x)-\lambda\int_0^\infty K(x,y)q(y)dy\}dx}{\{\int_0^\infty q(x)f(x)dx\}^2},$$

in which $q(x)$ is any bounded real well-behaved function, reaches its minimum for $q=q_0$, provided that either $\lambda\int_0^\infty K(x,y)dy\leqq 1$ for any fixed x, or $\lambda<\lambda_0$, when λ_0 is the limit, as $b\to+\infty$, of the lowest eigenvalue of the integral equation

$$\phi(x)=\lambda'\int_0^b K(x,y)\phi(y)dy.$$

Since the author's purpose is to give a rigorous justification of the variational method of solution of certain special problems given in Marshak's paper, he does not attempt to prove his result under the most general conditions possible nor to make his assumptions mutually independent.

E. T. Copson (Dundee).

Waller, Ivar. On the theory of the diffusion and the slowing down of neutrons. I. The diffusion of neutrons of constant energy in an infinite medium. Ark. Mat. Astr. Fys. **34A**, no. 3, 9 pp. (1947).

In this paper the equation

$$(*)\quad \lambda\mathbf{n}\cdot\operatorname{grad} H(\mathbf{r},\mathbf{n})+H(\mathbf{r},\mathbf{n})-\kappa\int d\Omega_{n'}\mu(\Phi)H(\mathbf{r},\mathbf{n}')=\kappa Q(r)$$

governing the stationary distribution of neutrons is considered for the case of an infinite isotropic homogeneous medium. In equation (*), λ and κ are constants, Φ is the angle between the directions of the unit vectors $\mathbf{n}$ and $\mathbf{n}'$ and $Q(r)$ is a known function of r; furthermore, $\mu(\Phi)$ is a "phase function" which is assumed to be given as a series in Legendre polynomials of the form

$$(**)\qquad \mu(\Phi)=(4\pi)^{-1}\sum_{n=0}^{\infty}(2n+1)C_nP_n(\cos\Phi).$$

Expressing $H(\mathbf{r},\mathbf{n})$ and $Q(r)$ in terms of their Fourier transforms according to

$$H(\mathbf{r},\mathbf{n})=(2\pi)^{-3}\int d\mathbf{u}e^{i\mathbf{u}\cdot\mathbf{r}}\sum_{n=0}^{\infty}(2n+1)k_n(u)P_n(\cos\theta),$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{n}$, and

$$q(u)=(4\pi/u)\int_0^\infty rQ(r)\sin ur\,dr,$$

the following system of equations among the functions k_n is derived:

$$\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta)(1+i\lambda u\cos\theta-\kappa C_n)k_n(u)=\kappa q(u),\qquad n=0,1,\cdots.$$

For the case when the expansion (**) for $\mu(\Phi)$ contains only a finite number of terms, the solution for $k_0(u)$ is obtained as a continued fraction and the k_n's are then derived successively from a recursion formula. *S. Chandrasekhar.*

Waller, Ivar. On the theory of the diffusion and the slowing down of neutrons. II. The energy distribution of neutrons slowed down by elastic impacts. Ark. Mat. Astr. Fys. **34A**, no. 4, 9 pp. (1947).

In this paper the stationary energy distribution of neutrons in an infinite isotropic homogeneous medium with K different species of atoms and slowed down by elastic impacts is considered. The basic system of equations to be considered is

$$(*)\quad g_n(x)=\kappa\sum_{k=1}^{K}\beta_k\int_{x-a_k}^{x}g_{n-1}(x')dx'+\delta_{n,0}g_0(x),\qquad n=1,2,\cdots,$$

where a_k, β_k $(k=1,\cdots,K)$ and κ are assigned constants, $g_0(x)$ is a known function and $\delta_{n,0}$ is the usual Kronecker symbol.

If f_n denotes the Fourier transform of g_n, then the solution of the system of equations (*) is shown to be expressible in the form

$$f_n(u)=\left[\kappa\sum_{k=1}^{K}\beta_k\frac{1-e^{-ia_ku}}{iu}\right]^n f_0(u).$$

The case when f_0 is a constant (i.e., when g_0 is of the nature of Dirac's δ-function) is considered in some detail. For this case the f_n's can be inverted to give explicit formulae for the functions g_n. Expressions giving the stationary energy distribution for all the neutrons are also derived for this case. *S. Chandrasekhar* (Williams Bay, Wis.).

Waller, I. On the theory of the diffusion and the slowing down of neutrons. III. The space energy distribution of neutrons in a moderator. Ark. Mat. Astr. Fys. **34A**, no. 5, 11 pp. (1947).

In this paper the complete stationary distribution of neutrons in an infinite isotropic medium with a spherically symmetric distribution of sources at the center is considered. It is assumed that the neutrons are slowed down by elastic impacts with an atom of a particular kind and further that the scattering by the atoms is isotropic in the center of gravity system of the neutron and the atom. The method used for solving the appropriate "equation of transfer" is a generalization of the one described in an earlier paper [see the second preceding review]. Approximations suitable for low energies are derived. *S. Chandrasekhar.*

Verde, M., and Wick, G. C. Some stationary distributions of neutrons in an infinite medium. Physical Rev. (2) **71**, 852–864 (1947).

This paper deals with the integro-differential equation governing the diffusion and slowing down of neutrons in an infinite homogeneous medium. The cases considered include those of a constant mean free path and a point source of neutrons and a variable mean free path proportional to a power of the velocity also with a point source of neutrons. The methods used have some similarity with those recently developed by I. Waller [see the three preceding reviews]. *S. Chandrasekhar* (Williams Bay, Wis.).

Functional Analysis, Ergodic Theory

Kaplansky, Irving. Lattices of continuous functions. Bull. Amer. Math. Soc. **53**, 617–623 (1947).

Si X est un espace compact (séparé) et $C(X)$ l'ensemble des fonctions réelles continues dans X, l'auteur montre que la structure d'ordre de $C(X)$ ($f\leqq g$ signifiant $f(x)\leqq g(x)$ pour tout $x\in X$) caractérise entièrement l'espace X: s'il existe un isomorphisme de la structure d'ordre de $C(X)$ sur celle de $C(Y)$, X et Y sont homéomorphes. La méthode de démonstration consiste à associer à un point $x\in X$ l'ensemble des "idéaux premiers" P dans le "lattice" $C(X)$ tels que $f\in P$ et $g(x)<f(x)$ entraîne $g\in P$. L'auteur montre que son théorème implique des résultats antérieurs analogues, où on considérait sur $C(X)$, soit sa structure d'anneau, soit sa structure d'espace de Banach [cf. M. H. Stone, Trans.

Amer. Math. Soc. **41**, 375–481 (1937); I. Gelfand et A. N. Kolmogoroff, C. R. (Doklady) Acad. Sci. URSS (N.S.) **22**, 11–15 (1939)]. Enfin, il prouve que toute décomposition de $C(X)$ en produits de deux lattices correspond à une décomposition de X en "somme topologique" de deux espaces compacts. *J. Dieudonné* (Nancy).

Gomes, Ruy Luís. Sur la notion de fonctionnelle. Portugaliae Math. **5**, 202–206 (1946).

If S is a space and $f(x)$ is a real functional on S, $f(x)$ is completely determined by the sets of points in S: $H(\lambda)=E[x; f(x)\geqq\lambda]$. This fact leads the author to define a functional more generally as a mapping $\lambda\to H(\lambda)$, where $H(\lambda)$ takes its values in some lattice and is a nonincreasing function of λ. The author proves for such functions $H(\lambda)$ the existence of the limits $H(\lambda-0)$, $H(\lambda+0)$ and develops properties which are the analogues of those which hold for monotone functions. By definition, a functional assumes the value λ if $H(\lambda-0)\neq H(\lambda+0)$. Thus the values assumed by the functional are precisely the points of discontinuity of $H(\lambda)$. *E. R. Lorch* (New York, N. Y.).

Hyers, D. H., and Ulam, S. M. Approximate isometries of the space of continuous functions. Ann. of Math. (2) **48**, 285–289 (1947).

Let X, X' be two compact metric spaces and $B(X)$, $B(X')$ the Banach spaces of real continuous functions over X, X' with the usual norm. The authors prove that, if there exists a homeomorphism T of $B(X)$ onto $B(X')$ which is an ϵ-isometry, i.e., such that $|\|T(f)-T(g)\|-\|f-g\||<\epsilon$ for all $f, g\in B(X)$, then there exists a true isometry U of $B(X)$ onto $B(X')$ such that $\|U(f)-T(f)\|\leqq 21\epsilon$ for all f in $B(X)$. The method used is to show that the existence of T implies the existence of a homeomorphism of X onto X'. This is accomplished by a study of the set A of all $f\in B(X)$ such that $|f(x)-b|\leqq a$ for a fixed $x\in X$, real number b and nonnegative real number a. It is shown that $T(A)\subset A'$, where A' is the set of all $f'\in B(X')$ such that $|f'(x')-c|\leqq a+3\epsilon/2$ and x' depends only on x (not on a or b). *S. B. Myers* (Ann Arbor, Mich.).

Taylor, Angus E. A geometric theorem and its application to biorthogonal systems. Bull. Amer. Math. Soc. **53**, 614–616 (1947).

Let S be a closed and bounded point set in Euclidean n-space E_n and let O be any point in E_n. For each set of n points $P_1, \cdots, P_n$ in S let $V(P_1, \cdots, P_n)$ denote the volume of the parallelepiped determined by the vectors $OP_1, \cdots OP_n$. The author shows that any set of points $P_1, \cdots, P_n$ for which V takes on its least upper bound has the additional property that for each $i=1, \cdots, n$ the point set $S\cup O$ lies entirely on and to one side of the unique $(n-1)$-dimensional hyperplane which passes through P_i and is parallel to the unique $(n-1)$-dimensional hyperplane through the points $O, P_1, \cdots, P_{i-1}, P_{i+1}, \cdots, P_n$. As an application he proves that for every finite dimensional subspace M of a normed linear space X there exists a basis $x_1, \cdots, x_n$ for M and a set $f_1, \cdots, f_n$ of continuous linear functionals on X such that $f_i(x_j)=\delta_i^j$ and $\|x_i\|=\|f_i\|=1$ for $i, j=1, \cdots, n$. *G. W. Mackey* (Cambridge, Mass.).

Livshitz, M. S. On a certain class of linear operators in Hilbert space. Rec. Math. [Mat. Sbornik] N.S. **19(61)**, 239–262 (1946). (Russian. English summary)

The author studies certain linear extensions of a closed linear isometric operator V with deficiency-index (1, 1), and also of the corresponding Hermitian operator A of like deficiency-index, obtained from V by the Cayley transformation under the condition (D) that the range of $V-I$ is everywhere dense. There is no loss of generality in supposing that V and A (when it exists) are simple (that is, irreducible). Let g, g' be arbitrary unit vectors orthogonal respectively to the domain and to the range of V. For $|\zeta|<1$ the operator $V_\zeta=(V-\zeta)(I-\bar{\zeta}V)^{-1}$ is of the same kind as V. While the vector g_ζ is not uniquely determined, the function $W(\zeta)=\zeta(g_\zeta, g')/(g_\zeta, g)$ obviously is. This function, called the characteristic function of V, is analytic interior to the unit circle; and $|W(\zeta)|\leqq 1$ there. Subject to these conditions, it may be prescribed arbitrarily. Two operators V_1, V_2 have the same characteristic function if and only if they are unitary-equivalent, in the sense that V_1, g_1, g_1' go over into V_2, g_2, g_2' by a unitary transformation of the basic space. The function $W_0(\lambda)=-W(\zeta)$, where $\zeta=(\lambda+i)(\lambda-i)^{-1}$, is taken as the characteristic function of the corresponding operator A (when the latter exists). This function is analytic in the upper half-plane, $|W_0(\lambda)|\leqq 1$ there, and (as a reflection of the condition (D) above) there is a real α such that $\lim_{|\lambda|\to\infty}\lambda[W_0(\lambda)-e^{i\alpha}]=0$ when $-\pi+\epsilon\leqq\arg\lambda\leqq\pi-\epsilon$. Subject to these conditions, it may be prescribed arbitrarily.

A linear extension T_a of V carrying g into ag' is called quasi-unitary. There is no loss of generality in supposing that $|a|\leqq 1$, $|T_a|\leqq 1$. When V has a Cayley transform A so does T_a. Such transforms are characterized as the linear operators B distinct from A and A^* such that $A\subset B\subset A^*$; they are called quasi-Hermitian operators. The following spectral properties are proved for T_a: the spectral points interior to the unit circle are all characteristic values and are precisely the roots of $W(\zeta)=a$; a point ζ_0 on the unit circle is in the limit-spectrum of T_a (that is, f_n exists so that $|f_n|=1$, $T_af_n-\zeta_0f_n\to 0$) if and only if it is in the limit-spectrum of V, and also if and only if it is a singular point of W in the sense that W can be continued analytically over a neighborhood of ζ_0, if at all, only by taking $|W(\zeta_0)|<1$. Analogous results hold for quasi-Hermitian operators. The Hermitian differential operator A defined by putting $Af=-idf/dx$ when $f(0)=f(a)=0$ is characterized up to unitary-equivalence by the fact that it has at least one quasi-Hermitian extension whose spectrum contains no characteristic values and no limit-spectrum. The final section of the paper sketches the operational calculus which results from putting $V_2=\phi(V_1)$ when $W_2(\zeta)=\phi(W_1(\zeta))$. Applications are indicated in some interesting special cases, including that of the operator $A=c(m^2c^2I-h^2d^2/dx^2)^{\frac{1}{2}}$ considered in quantum theory. *M. H. Stone* (Chicago, Ill.).

Nikodym, Otton Martin. Tribus et lieux attachés à une classe ordonnée de sous-espaces d'un espace de Hilbert-Hermite. C. R. Acad. Sci. Paris **224**, 628–630 (1947).

This note describes a theory of families of subspaces of a generalized complex Euclidean space, continuing an earlier publication [same C. R. **224**, 522–524 (1947); these Rev. **8**, 386]. The principal results appear to be of a kind easily deducible from the standard theory of commutative families of operators (here, projection operators) and their operational calculus. *M. H. Stone* (Chicago, Ill.).

Nikodym, Otton Martin. Système général de coordonnées dans un espace séparable de Hilbert-Hermite. C. R. Acad. Sci. Paris **224**, 778–780 (1947).

This note continues two earlier publications [cf. the preceding review]. Here the author announces the definition

of an integral for vector-valued functions of subspaces of a generalized complex Euclidean space, and the derivation of integral formulas connecting a fixed vector with its projections on the subspaces of a given family. *M. H. Stone.*

de Sz. Nagy, Béla. Perturbations des transformations autoadjointes dans l'espace de Hilbert. Comment. Math. Helv. **19**, 347–366 (1947).

This paper is an extension of the work of F. Rellich [Math. Ann. **113**, 600–619 (1936), 677–685 (1936); **116**, 555–570 (1939); **117**, 356–382 (1940); **118**, 462–484 (1942); these Rev. **2**, 105; **6**, 71] on the perturbation of the spectrum of self-adjoint operators $A(\epsilon)$ which depend on a parameter ϵ. A typical result (partial statement of theorem I) is as follows. Let $A_0, A_1, \cdots$ be symmetric linear transformations of $\mathfrak{H}$ with common domain $\mathfrak{D}$; further, let A_0 be self-adjoint and $\|A_k f\| \leqq p^{k-1}(a\|f\|+b\|A_0 f\|)$ $(k=1, 2, \cdots)$ for all $f \in \mathfrak{D}$, p, a, b being nonnegative real constants. (a) The series $A_0+\epsilon A_1+\epsilon^2 A_2+\cdots$ converges for $|\epsilon|<1/p$ and has for its sum a symmetric transformation $A(\epsilon)$ whose domain is $\mathfrak{D}$. At least for $|\epsilon|<1/(p+b)$, $A(\epsilon)$ is self-adjoint. (b) Suppose that the interval $\Delta=[\mu_1, \mu_2]$ contains an isolated part of the spectrum of A_0, or, more precisely, that there exist two open intervals of length d, $0<d\leqq\mu_2-\mu_1$, having centers μ_1 and μ_2 and not containing any point of the spectrum of A_0. Then for $|\epsilon|<1/(p+a)$, the intervals of length $d(\epsilon)=d(1-|\epsilon|\alpha/(p+a))$ about the centers μ_1 and μ_2 do not contain any points of the spectrum of $A(\epsilon)$; here $\alpha=2a/d+2b(1+|\lambda_0|/d+\frac{1}{2}\delta/d)$, $\lambda_0=\frac{1}{2}(\mu_1+\mu_2)$, $\delta=\mu_2-\mu_1-d$. This theorem is then used to obtain similar results on the effects on the spectrum of $A(\epsilon)$ of isolated proper values of A_0 of simple, finite and infinite multiplicity.

R. A. Leibler (Princeton, N. J.).

Alonso, Marcelo. Some properties of Hermitian operators. Revista Soc. Cubana Ci. Fís. Mat. **2**, 87–91 (1946). (Spanish)

The author discusses some questions concerning the operators used in quantum mechanics. It is shown that, if an operator has real eigenvalues, and if its eigenfunctions form a complete set of orthogonal functions, the operator is Hermitian. It is also shown that, if the operators A and B are Hermitian, so also is the operator $AB+BA$.

L. A. MacColl (New York, N. Y.).

Wintner, Aurel. The unboundedness of quantum-mechanical matrices. Physical Rev. (2) **71**, 738–739 (1947).

The author shows that, if two Hermitian infinite matrices P and Q satisfy Heisenberg's relation, taken in the form $QP-PQ=\frac{1}{2}iE$, where E is the unit matrix, they cannot both be bounded. *F. Smithies* (Cambridge, England).

Fantappiè, Luigi. L'indicatrice proiettiva dei funzionali lineari e i prodotti funzionali proiettivi. Ann. Mat. Pura Appl. (4) **22**, 181–289 (1943).

L'auteur poursuit une étude des fonctionnelles linéaires dont les arguments sont des fonctions analytiques de n variables complexes. Plusieurs des resultats et formules donnés se trouvent déjà indiqués pour $n=2$ dans ses mémoires antérieures. Il appelle indicatrice de la fonctionnelle F la fonction $F[y(t_1, \cdots, t_n)]$ avec les choix particuliers suivants: (a) $y=1/(\alpha_1-t_1)\cdots(\alpha_n-t_n)$: indicatrice semi-symétrique $u(\alpha_1, \cdots, \alpha_n)$; (b) $y=1/(1-\alpha_1 t_1)\cdots(1-\alpha_n t_n)$: indicatrice symétrique $v(\alpha_1, \cdots, \alpha_n)$; (c) $y=1/(1+\alpha_1 t_1+\cdots+\alpha_n t_n)$: indicatrice projective $p(\alpha_1, \cdots, \alpha_n)$. La fonction

$$\chi_n=\sum_{r_1, \cdots, r_n}(-1)^r\frac{r_1!r_2!\cdots r_n!}{r!}z_1^{r_1}z_2^{r_2}\cdots z_n^{r_n}, \qquad r=r_1+\cdots+r_n,$$

permet le passage de (b) à (c). Si l'on pose:

$$\chi_n(z_1t_1^*, \cdots, z_nt_n^*)y(t_1^*, \cdots, t_n^*)$$
$$=\frac{1}{(2\pi i)^n}\int_{C_1}dt_1\cdots\int_{C_n}dt_n\frac{1}{t_1\cdots t_n}\chi_n\left(\frac{z_1}{t_1}, \cdots, \frac{z_n}{t_n}\right)y(t_1, \cdots, t_n),$$

où les contours C_i entourent $t_i=0$, le produit de composition commutatif

$$p^*y^*=y^*p^*=p(\alpha_1^*, \cdots, \alpha_n^*)\chi_n(\alpha_1^*t_1^*, \cdots, \alpha_n^*t_n^*)y(t_1^*, \cdots, t_n^*)$$

est appelé produit fonctionnel projectif, et l'on a $F[y]=p^*y^*$.

Le produit p^*y^* est invariant quand on fait subir aux variables z_n d'une des fonctions une substitution linéaire à coefficients constants, celles de l'autre subissant la substitution duale: p^*y^* est ainsi une fonction de 2 vecteurs, de variances différentes. Lorsque y est un produit ordinaire de facteurs y_k, p^*y^* est calculé au moyen des produits $p^{*\prime}y_k^*$; p', moyenne projective de p, étant définie à partir de p par des quadratures simples.

La fin du mémoire est consacrée à l'étude des fonctionnelles F dont l'indicatrice projective p est rationnelle. L'auteur en fait une application à l'étude du problème de Cauchy pour les équations aux dérivées partielles à coefficients constants, d'ordre quelconque. Quelques questions se posent quant aux singularités des fonctions analytiques obtenues quand on se donne F effectivement; leur approfondissement permettrait, semble-t-il, d'apprécier les progrès que l'étude générale des fonctionnelles de l'auteur apporte à la théorie des fonctions analytiques.

P. Lelong (Lille).

Hartman, Philip. On the ergodic theorems. Amer. J. Math. **69**, 193–199 (1947).

This paper gives a new method of proof of the Birkhoff and mean ergodic theorems. The decomposition of image sets which appears in the proof of the Birkhoff theorem is avoided by use of a lemma related to the fundamental theorem of the integral calculus and due to F. Riesz. There results an improved form of the fundamental inequality on which the proof of Birkhoff's theorem depends. The mean ergodic theorem is a consequence of this inequality and the Birkhoff theorem. *G. A. Hedlund* (Charlottesville, Va.).

Barbachine, E. Sur la classification des multiplicités intégrales d'un système d'équations en différentielles totales. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 279–282 (1947).

Beginning with a system of total differential equations, the author indicates how one is led to consideration of a transformation group acting on a compact subset X of n-dimensional Euclidean space, where the group itself is the group R_n of vectors in n-dimensional space. If x_0 is a point of X and W denotes the subgroup of R_n which leaves x_0 invariant, the factor group R_n/W is the direct sum of r groups each of which is isomorphic to the additive group of reals and k groups each of which is isomorphic to the multiplicative group of complex numbers with modulus 1. The orbit of x_0 is then denoted by $Z_{r,k}$ and the group W by $W_{r,k}$. If $\mathbf{A}$ denotes a unit vector of R_n, an ordering is defined in R_n by the assertion that $\mathbf{B}<\mathbf{C}$ provided $\mathbf{B}\cdot\mathbf{A}<\mathbf{C}\cdot\mathbf{A}$. This establishes a connection with an earlier paper by the author [same C. R. (N.S.) **51**, 3–5 (1946); these Rev. **8**, 34] and enables him to define orbits which are recurrent, totally stable, etc., with respect to the direction $\mathbf{A}$. The analysis of an orbit is particularly simple if it is such that $Z_{r,k}$ is the topological image of $W_{r,k}$, in which case it is said to be

normal. It is asserted that the following orbits are normal: (a) all orbits of type $Z_{k,k}$; (b) the n-dimensional orbits; (c) $(n-1)$-dimensional orbits which are of type $Z_{k,n-2}$.

G. A. Hedlund (Charlottesville, Va.).

Mayer, A. G. Sur un problème de Birkhoff. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 473–475 (1947).

The author outlines the construction of a dynamical system with the properties that the phase space is a compact three-dimensional manifold and the ordinal number attached to the central motions (by repeated passage to the non-wandering motions) is an arbitrary preassigned positive integer. The construction of such a dynamical system solves for the three-dimensional case a problem proposed by G. D. Birkhoff [Jber. Deutsch. Math. Verein **38**, 1–16 (1929); Science **94**, 598–600 (1941); these Rev. **3**, 279].

W. H. Gottschalk (Philadelphia, Pa.).

Mayer, A. G. Sur les trajectoires dans l'espace à trois dimensions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 579–581 (1947).

The author outlines the construction of a dynamical system with the properties that the phase space is three-dimensional Euclidean space and the ordinal number attached to the central motions is not less than an arbitrary preassigned transfinite ordinal number of the first class. [Cf. the preceding review.] *W. H. Gottschalk.*

Calculus of Variations

Rosenblatt, Alfred. Some reciprocity theorems in the calculus of variations. Revista Ci., Lima **48**, 317–355 (1946). (English. Spanish summary)

The author considers an extremal curve E between two points P_1, P_2 in the plane lying on transversal curves C_1, C_2, respectively. He demonstrates a reciprocity theorem between the families of extremals E_1 and E_2 which pass through P_1 and P_2. This theorem is then generalized to the case of an extremal in space, to the case of transversal surfaces as well as transversal curves and finally to the case in which the extremals are restricted to lie on a given surface.

J. E. Wilkins, Jr. (Buffalo, N. Y.).

Sakellariou, Nilos. Sur le problème adjoint au calcul des variations. Bull. Soc. Math. Grèce **21**, 53–57 (1941).

For a regular variational problem in parametric form of the type

$$\delta\int_G F(p^{ik})dudv=0, \quad p^{ik}=(x_i, x_k)/(u, v),$$

the author expresses the transversality condition as a condition on the surface conjugate to a given extremal surface. This paper may be considered as a continuation of a previous one by the author [Monatsh. Math. Phys. **48**, 314–321 (1939); these Rev. **1**, 78]. *W. T. Reid.*

Van Hove, Léon. Sur une classe de problèmes de valeurs propres et une modification de la notion de système complet de fonctions orthogonales. Bull. Soc. Roy. Sci. Liége **15**, 306–314 (1946).

The author points out that the integral

$$I(x, \bar{x})=\int_D (a_{ij}x_ix_j+\bar{a}_{ij}\bar{x}_i\bar{x}_j+b_{ij}x_i\bar{x}_j+a_ix_ix+\bar{a}_i\bar{x}_i\bar{x} +b_ix_i\bar{x}+\bar{b}_i\bar{x}_ix+ax^2+\bar{a}\bar{x}^2+bx\bar{x})dt,$$

in which $x_i=\partial x/\partial t_i$ $(i=1, \cdots, n)$, $dt=dt_1\cdots dt_n$, and the bar denotes complex conjugate, is always real and supplies a generalization of the classical integral with a Hermitian integrand when $a_{ij}=0$, $a_i=0$, $a=0$. If x and $\bar{x}$ are replaced by distinct functions u and v, the first variation of I may be written in the form

$$\delta I(u, v)=-\int_D (L_1(u, v)\delta u+L_2(u, v)\delta v)dt,$$

in which L_1 and L_2 are linear homogeneous differential expressions of the second order. The author then considers the equations $L_1(u, v)+\lambda\phi(t)v=0$, $L_2(u, v)+\lambda\phi(t)u=0$, in which $\phi(t)>0$ on D and u and v vanish on the boundary of D. He shows that the characteristic values and functions for this system possess essentially the same properties that they possess in the classical case provided that there exist two numbers μ_0 and μ_1 such that $I(x, \bar{x})\geqq\int_D(\mu_0x_i\bar{x}_i+\mu_1x\bar{x})dt$.

J. E. Wilkins, Jr. (Buffalo, N. Y.).

Belgodère, Paul. Interprétation, par l'indicatrice des aires, de la condition d'Euler pour l'extremum, dans un problème variationnel. C. R. Acad. Sci. Paris **224**, 1203–1205 (1947).

The author extends results obtained in an earlier paper [same C. R. **218**, 739–740 (1944); these Rev. **7**, 385] to the case of a double integral whose integrand function $g(x, y, z, p, q)$ is a function of x, y, z in addition to p, q.

J. E. Wilkins, Jr. (Buffalo, N. Y.).

Wagner, V. On a sufficient condition in the problem of Lagrange for multiple integrals. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 479–482 (1946).

The author gives a definition of a geodesic field for a parametric problem of Lagrange for multiple integrals in the calculus of variations which generalizes that of Carathéodory [Acta Litt. Sci. Univ. Szeged **4**, 193–216 (1929)] for ordinary parametric problems with no side conditions. He then shows that a surface surely furnishes an extremum when it can be imbedded in a geodesic field which satisfies the Weierstrass condition. *J. E. Wilkins, Jr.*

Wagner, V. On a geometric interpretation of the extremal surfaces in the problem of Lagrange for multiple integrals. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 87–90 (1947).

An extremal surface possessing a set of multipliers having the property that the multiplier for the integrand function is not zero is either normal (in case the multipliers are unique) or weakly anormal. The author shows that in a suitable geometry such extremal surfaces may be regarded as having zero mean curvature. *J. E. Wilkins, Jr.*

Theory of Probability

Gleissberg, W. Bedingungen für die Anordnung zufälliger Fehler. Rev. Fac. Sci. Univ. Istanbul (A) **12**, 107–126 (1947). (German. Turkish and English summaries)

By elementary computations the author finds that in a random arrangement of n white and n black objects, the mean length of a run of like elements is $2n/(n+1)$; the purpose of these elementary computations is a supposedly new test of randomness. [Similar tests based on runs are much used. For literature cf. S. S. Wilks, Mathematical Statistics, Princeton University Press, 1943, chap. X; these Rev. **5**, 41.] *W. Feller* (Ithaca, N. Y.).

Malmquist, Sten. A statistical problem connected with the counting of radioactive particles. Ann. Math. Statistics **18**, 255–264 (1947).

A familiar counter problem is as follows. Suppose that the distances between two consecutive events are mutually independent random variables with a common distribution $F(b)$. A "transformed" sequence of random events is formed by omitting as many of the original events as necessary to get a fixed minimum distance u between two consecutive events. The author shows how the distribution of the new distances can be computed by means of convolutions. Explicit formulas are derived in the case $F(b)=1-\exp(at)$ for the first and second transformed sequence. As an illustration an experiment with random sampling numbers is performed although it is not clear whether it is intended to verify the numerical calculations or the randomness of the (already tested) tables. *W. Feller* (Ithaca, N. Y.).

Forsythe, George E. On Nörlund summability of random variables to zero. Bull. Amer. Math. Soc. **53**, 302–313 (1947).

A typical result is the following. If N_q is a regular Nördlund summability method satisfying for all n the condition $q_n \leqq q_{n+1}$, then it is equivalent with the C_1 method with respect to summability in probability to 0, of normal families of independent real-valued symmetric random variables. *M. Kac* (Ithaca, N. Y.).

Linnik, U. V. On the accuracy of the approximation to the Gauss distribution by sums of independent variables. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **11**, 111–138 (1947). (Russian. English summary)

Strong estimates are obtained for the error term in the central limit theorem. Let $\{\mathfrak{x}_j\}$ be mutually independent symmetric random variables with vanishing expectations, second moments σ_j^2 and third absolute moments a_j. Let $s_n^2=\sum_1^n \sigma_j^2$ and $A_n=\sum_1^n a_j$. The Liapunov ratio is then $L_n=A_n s_n^{-3}$. The essential assumptions are that the σ_j^2 are bounded away from zero, $s_n^2=O(n)$ and $L_n \to 0$. Moreover, if $a_j^{(\epsilon)}$ is the third absolute moment of $\mathfrak{x}_j$ truncated at the points $\pm \epsilon s_n$ then $A_n^{-1}\sum_1^n a_j^{(\epsilon)}$ is to tend to 1 uniformly in ϵ. Finally $\Pr\{|\mathfrak{x}_j| \leqq L_n^{\eta}\}$ is to tend to 0 uniformly in $j \leqq n$, where η is a constant, $\frac{1}{2}<\eta<1$. If $F_n(x)$ is the distribution of $(\mathfrak{x}_1+\cdots+\mathfrak{x}_n)/s_n$ and $\Phi(x)$ the normalized Gaussian distribution, then in every bounded interval $|x|<M$

$$(1) \qquad |F_n(x)-\Phi(x)| \leqq L_n(2\pi)^{\frac{1}{2}} e^{-x^2/2}(1+\epsilon_n)$$

with $\epsilon_n \to 0$. This result is in a sense the best since for the symmetric Bernoulli distribution it is possible to exhibit a sequence of points x_n such that the inequality in (1) is reversed. On the other hand, it is shown that under the given conditions there always exists a sequence of equidistant values x_k with $\Delta x_k \to 0$ for which the left hand in (1) is $o(L_n)$. The theorems are generalized to unsymmetric variables $\mathfrak{x}_j$ for which the positive and negative part have same probability and contribute equally to σ_j^2. The proof is based on characteristic functions and uses a technique said to have been introduced by Vinogradov in additive number theory and based on smoothing of trigonometrical sums. *W. Feller* (Ithaca, N. Y.).

Linnik, U. V. On the accuracy of the approximation of the Gaussian distribution by sums of independent random variables. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 571–573 (1947).

This is almost identical with the English summary of the paper reviewed above. *W. Feller* (Ithaca, N. Y.).

Cramér, Harald. Problems in probability theory. Ann. Math. Statistics **18**, 165–193 (1947).

Lecture at the Princeton Bicentennial Conference.

Bayley, G. V., and Hammersley, J. M. The "effective" number of independent observations in an autocorrelated time series. Suppl. J. Roy. Statist. Soc. **8**, 184–197 (1946).

The author remarks that if $\cdots, x_0, x_1, \cdots$ are the chance variables of a stationary stochastic process the equation var. $(1/n)\sum_{j=1}^n x_j = (\text{var. } x_1)/n$ is not necessarily true. From one point of view the actual denominator n_b on the right in this equation is the "effective number of independent observations." If n measurements are made of a continuous parameter process over a time interval of length T, the author measures the efficiency of the computation by the ratio n_b/n, and this suggests a criterion for the choice of n, that is, for the frequency of measurement. The quantities involved are calculated for a process with covariance function of the form $e^{-\mu|s|}\cos \lambda s$ (where λ and μ are constant parameters). The corresponding work, leading to different efficiency values, is also carried out with the sample average replaced by the sample variance. *J. L. Doob.*

Onicescu, Octav, et Mihoc, Gh. Comportement asymptotique des chaînes à liaisons complètes. Disquisit. Math. Phys. **1**, 61–62 (1940).

The authors sketch a proof that if $x_1, x_2, \cdots$ are the chance variables of a stationary chain "à liaisons complètes" [cf. the authors' "La dépendance statistique," Actual. Sci. Ind., no. 503, Paris, 1937] where the x_j's take on only two values, then under rather general conditions $x_1+\cdots+x_n$ is asymptotically normally distributed for large n. *J. L. Doob* (Urbana, Ill.).

Greenwood, Major. The statistical study of infectious diseases. J. Roy. Statist. Soc. (N.S.) **109**, 85–103; discussion, 103–110 (1946).

The main part of the paper does not deal with problems of mathematical interest. Appendix 1 [pp. 97–102] considers the problem of n points taken at random on an interval of length L. The first four moments of the distribution of the length of the partial intervals are calculated. Pearson curves are fitted to this distribution. The probability distribution of the sum of squares of the $n+1$ intervals is also derived. *W. Feller* (Ithaca, N. Y.).

Domb, C. The problem of random intervals on a line. Proc. Cambridge Philos. Soc. **43**, 329–341 (1947).

Consider random intervals on the t-axis with a constant length, whose left endpoints are random events distributed according to a Poisson distribution. Let $W(x, y)$ be the probability that the measure of the subset of an arbitrary interval $(t, t+y)$ which is covered by the random intervals does not exceed x. It is shown that $W(x, y)$ satisfies an integral equation of the convolution type. The author obtains an explicit formula for a Laplace transform. From it he is able to derive many results which have recently occupied several authors. In particular, he derives an expansion of $W(x, y)$ according to the number of events in the interval, approximations to $W(x, y)$ for large y, moments of the distribution, the distribution of gaps. The method yields analogous results also for the case of random intervals on a circle. *W. Feller* (Ithaca, N. Y.).

Walters, A. G. The distribution of projected areas of fragments. Proc. Cambridge Philos. Soc. **43**, 342–347 (1947).

A cylinder of radius r and length l lies with its axis in the x-axis and its midpoint at the origin. For a direction of attack at an angle θ to the x-axis the "aspect area" of the cylinder is $A=\pi r^2\cos\theta+2rl\sin\theta$. Assuming that the directions have uniform distribution the author derives the distribution and the variance of A. The same problem is solved for a rectangular parallelepiped. *W. Feller.*

Moran, P. A. P. Random associations on a lattice. Proc. Cambridge Philos. Soc. **43**, 321–328 (1947).

In a rectangular two-dimensional lattice points are black or white in accordance with the Bernoulli scheme. Let N be the number of black-white joints. The first four moments of N are calculated. Using Bernstein's form of the central limit theorem for dependent variables it is shown that the distribution of N approaches normality as the size of the rectangles increases. The result is generalized to three dimensions. *W. Feller* (Ithaca, N. Y.).

Robbins, H. E. Acknowledgement of priority. Ann. Math. Statistics **18**, 297 (1947).

The theorem on page 72 of the author's paper in the same Ann. **15**, 70–74 (1944) [these Rev. **6**, 5] had been anticipated by Kolmogoroff [Grundbegriffe der Wahrscheinlichkeitsrechnung, Ergebnisse der Math., v. 2, no. 3, Springer, Berlin, 1933, p. 41].

Mathematical Statistics

***Finney, D. J. Probit Analysis. A Statistical Treatment of the Sigmoid Response Curve.** Cambridge, at the University Press; New York, The Macmillan Company, 1947. xiii+256 pp. $3.75.

This book is essentially a handbook for the statistical analysis of all-or-none response data. Commonly the probability of a positive response increases continuously with increasing values of the stimulus and can, by a transformation of the stimulus scale, be fitted by a cumulative normal curve. Computational methods appropriate to a wide variety of practical problems are presented in adequate detail for the practitioner; the mathematical theory is presented in a brief appendix. The tables provided are more extensive than any elsewhere available. Six pages of references are given, covering both theory and applications to special problems. *C. P. Winsor* (Baltimore, Md.).

Skellam, J. G. The frequency distribution of the difference between two Poisson variates belonging to different populations. J. Roy. Statist. Soc. (N.S.) **109**, 296 (1946).

Let X_j be two independent Poisson variables with means m_j $(j=1, 2)$. The generating function of $p_r=\Pr\{X_1-X_2=r\}$ is $G(t)=\exp(-m_1-m_2+m_1t+m_2t^{-1})$. From this the author derives $p_r=\exp(-m_1-m_2)(m_1/m_2)^rI_r(2m_1^{\frac{1}{2}}m_2^{\frac{1}{2}})$, where $I_n(z)$ is the familiar Bessel function. *W. Feller.*

Radhakrishna Rao, C. Note on a problem of Ragnar Frisch. Econometrica **15**, 245–249 (1947).

The author is concerned with conditions for the regression of one variable on another to be linear. A general lemma is established: if $\theta(t_1, t_2)$ is the characteristic function of x_1 and x_2, and $E(x_1)=E(x_2)=0$, then a necessary and sufficient condition that the regression of x_1 on x_2 is linear is that $[\partial\theta(t_1, t_2)/\partial t_1]_{t_1=0}=b_{12}d\theta(0, t_2)/dt_2$. This result is applied to two special cases. (I) If $w_1, \cdots, w_k; y_1, \cdots, y_m; z_1, \cdots, z_n$ are independent observations from three populations, all having mean zero, and if $x_1=w_1+\cdots+w_k+y_1+\cdots+y_m$ and $x_2=w_1+\cdots+w_k+z_1+\cdots+z_n$, then a necessary and sufficient condition that the regression of x_1 on x_2 is linear is that $\phi=\eta^\lambda$, where $\phi(t)$ and $\eta(t)$ are the characteristic functions of w and z, respectively, and λ is a constant. (II) If $x_1=a\xi+\alpha$ and $x_2=b\xi+\beta$, where a and b are constants, ξ, α and β have zero means, β is independent of ξ and α, and $E(\alpha/\xi)=0$ for all ξ, then a necessary and sufficient condition that the regression of x_1 on x_2 is linear for all a and b is that $\phi=\exp(ct^d)$, $\eta=\exp(c't^d)$, where ϕ and η are characteristic functions of ξ and β, respectively, and $1<d\leqq 2$. The last theorem is the problem of Frisch. When $d<2$, the variances of ξ and β are infinite; when $d=2$, ξ and β are normally distributed. *K. L. Arrow* (Chicago, Ill.)

Fréchet, M. Definition of the probable deviation. Ann. Math. Statistics **18**, 288–290 (1947).

Fréchet, M. A general method of constructing correlation indices. Proc. Math. Phys. Soc. Egypt **3**, 13–20 (1946).

Theoretical desiderata for an index of correlation which is to vanish for all one-valued functions $y=f(x)$ can be satisfied by indices based on a notion of distance between distribution functions. Let (G_1, G_2) be the maximum of $G_1(x)-G_2(x)$ over the common points of continuity of G_1 and G_2. Let F be the unconditional distribution of y, F_x the conditional distribution of y given x, and Φ_x the distribution concentrated at the median of F_x. Then

$$H=\frac{\overline{(F_x, F)}}{\overline{(\Phi_x, F_x)}+\overline{(F_x, F)}}$$

is such an index, the bars representing averages over x. The sample analogue of this population index is defined and computed for an example. It is pointed out that, if each class of x contains only one y, then the sample index is always equal to 1 [cf. the following review]. [See also Fréchet, Econometrica **15**, 1–30 (1947); these Rev. **8**, 393.] *J. W. Tukey* (Princeton, N. J.).

Simaika, J. B. Note on M. Fréchet index of correlation. Proc. Math. Phys. Soc. Egypt **3**, 21–22 (1946).

It is pointed out that estimating Frechét's population index from a sample [cf. the preceding review] might well be done by other processes than calculating the sample index. *J. W. Tukey* (Princeton, N. J.).

Taylor, Erwin K. Tables for the determination of the significance of skewness and of the significance of the difference in the skewness of two independent distributions. Psychometrika **12**, 111–125 (1947).

Penrose, L. S. Some notes on discrimination. Ann. Eugenics **13**, 228–237 (1947).

The problem of combining measurements of different kinds to discriminate between different populations, that is, to assign unknown individuals to one or the other population on the basis of samples from the populations, was first treated by R. A. Fisher [Ann. Eugenics **7**, 179–188 (1936)] who studied the case where the variances and covariances of both populations are the same. The author proposes to simplify this process by calculating two sum-

mary measures, the "size" and "shape," which are asymptotically uncorrelated, and then to use a linear compound of these two summary measures for discrimination. If all intercorrelations are equal this method is 100% efficient in large samples. The individual measures are standardized and then weighted, equally for "size" and in accordance with population differences for "shape." The value of absolute deviations from a suitably chosen value in discriminating between two populations with different variances is pointed out. *J. W. Tukey* (Princeton, N. J.).

Smith, Cedric A. B. Some examples of discrimination. Ann. Eugenics **13**, 272–282 (1947).

The author applies the likelihood ratio principle to the determination of the best discriminator for two normal populations [cf. the preceding review] with different variances and covariances. The result is a quadratic expression. By using observed variances and covariances as estimates of the population values, the quadratic discriminator can be applied to observational data. Examples are given.
J. W. Tukey (Princeton, N. J.).

Vajda, S. Average sampling numbers from finite lots. Suppl. J. Roy. Statist. Soc. **8**, 198–201 (1946).

An exact combinatorial expression for the average sample size is obtained when sampling from lots of size N with M defectives so that the sampling ceases when either $C+1$ defectives are found or when it is apparent that n items will be drawn with fewer than $C+1$ defectives. Tables of the average sample size are given for $N=10$, $C=0, 1$.
A. M. Mood (Ames, Iowa).

Anderson, T. W. A note on a maximum-likelihood estimate. Econometrica **15**, 241–244 (1947).

The author discusses the problem of estimating the constant $\gamma=2\pi/\arccos(-\alpha/2\beta^{\frac{1}{2}})$ involved in the stochastic difference equation (*) $x_t+\alpha x_{t-1}+\beta x_{t-2}=u_t$, where the u_t's are mutually independent and have zero means; γ is the period of the solution of the homogeneous difference equation. He estimates α and β by the method of maximum likelihood, substituting these estimates in (*) to obtain an estimate of γ. The estimate thus obtained is consistent and asymptotically normal; the asymptotic variance is evaluated. As judged by four empirical series, the author's estimate of γ is somewhat more exact than estimates used by Kendall [Contributions to the Study of Oscillatory Time-Series, Cambridge University Press, 1946; these Rev. **8**, 471]. *J. L. Doob* (Urbana, Ill.).

Smith, John H. Estimation of linear functions of cell proportions. Ann. Math. Statistics **18**, 231–254 (1947).

Eisenhart, Churchill. The assumptions underlying the analysis of variance. Biometrics **3**, 1–21 (1947).

This is the first of a set of three expository papers [cf. the two following reviews] which critically discuss aspects of the analysis of variance for the benefit of those interested in the general applicability of this important statistical method. The present paper points out with clarity and emphasis the distinction between the two main types of application: (1) the "detection and estimation of fixed relations among the means of subsets of the universe of objects concerned" and (2) the "detection and estimation of components of variation associated with a composite population." The greater part of the paper is devoted to the statement and discussion of the meaning of the assumptions on which the mathematical theory leading to the significance tests is based. *C. C. Craig* (Ann Arbor, Mich.).

Cochran, W. G. Some consequences when the assumptions for the analysis of variance are not satisfied. Biometrics **3**, 22–38 (1947).

The second of this series of papers [cf. the preceding review] develops the subject stated in its title by discussing with illustrations what may be expected and what useful results may still be obtainable when the underlying assumptions for the theory fail in ways that do occur and are of importance in actual experimental situations.
C. C. Craig (Ann Arbor, Mich.).

Bartlett, M. S. The use of transformations. Biometrics **3**, 39–52 (1947).

In the third of this series of three papers [see the two preceding reviews] the author discusses the utility of the important transformations on the experimentally observed variable which have been used to make the underlying mathematical theory usefully applicable. Some of these are numerically illustrated. *C. C. Craig* (Ann Arbor, Mich.).

***Wald, Abraham. Sequential Analysis.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1947. xii+212 pp. $4.00.

The book consists of three parts and an appendix. Part I (General theory) develops the theory of the author's sequential probability ratio test for a statistical hypothesis. In part II (Application of the general theory to special cases) tests for the following parameters are discussed in detail: (1) mean of a binomial, (2) difference between means of two binomials, (3) mean of a normal distribution with known variance (both one-sided and two-sided tests), (4) standard deviation of a normal distribution. Part III (The problem of multi-valued decisions and estimation) is a short [twenty pages] very general discussion of the two problems of its title: formulations of the problems together with principles for their solutions.

With the exception of part III the book contains essentially the material published in two technical papers by the author [Ann. Math. Statistics **16**, 117–186 (1945); J. Amer. Statist. Assoc. **40**, 277–306 (1945); these Rev. **7**, 131, 132]. The material has been completely rewritten, however, so that the book may be read by people without extensive technical training. All the more difficult mathematical material has been put in the appendix. In fact, the author suggests that an elementary course in calculus should be sufficient preparation, but persons without a course in mathematical statistics would probably find the book rather difficult.

The appendix [sixty pages] contains proofs of the theorems required in the book proper. It is not a connected account; persons interested only in the mathematical aspects of the theory would prefer to read the first article referred to above.

The book is written in the author's usual completely lucid manner. It is highly recommended to all who wish to learn about sequential tests of hypotheses. The theory of sequential estimation is, of course, yet to be developed. The book may also be used as a text for undergraduates (without the appendix) or graduates (with the appendix), though here it is not all that could be desired because it contains no problems for students. *A. M. Mood.*

Blackwell, D., and Girshick, M. A. A lower bound for the variance of some unbiased sequential estimates. Ann. Math. Statistics **18**, 277–280 (1947).

Assume $W_k=x_1+\cdots+x_k$ is a sufficient estimator of the mean θ of a population with variance $\sigma^2(x)$ for samples of size k. It was shown by Blackwell [same Ann. **18**, 105–110 (1947); these Rev. **8**, 478] that $V=E(x_1|W, n)$ is an unbiased estimator of θ for sequential sampling, where n is the sample size. Here it is shown that $\sigma^2(V)\geqq\sigma^2(x)/E(n)$ and that this lower bound is attained when and only when the sample size is fixed in advance. *A. M. Mood.*

Harris, T. E. Note on differentiation under the expectation sign in the fundamental identity of sequential analysis. Ann. Math. Statistics **18**, 294–295 (1947).

The identity $E\{e^{it\sum_1^n x_i}[\phi(t)]^{-n}\}=1$ of sequential analysis may be differentiated k times with respect to t under the expectation sign when t is purely imaginary provided $E(x^k)$ and $E(n^k)$ are finite, and provided there exists a constant M such that $|Z_n|<M$ when $n>N$. *A. M. Mood.*

Krylov, V. The serial samples. Acta [Trudy] Univ. Asiae Mediae. Ser. V-a. Fasc. **25**, 24 pp. (1939). (Russian. English summary)

A serial sampling scheme is by definition one in which the experimental universe is divided into a number M of subsets ("series"), of which a certain number m are chosen for further study. The initial subdivision is fixed, once made, and is to this extent not part of the sampling process; e.g., the inhabitants of a country are groupéd by place of employment. The sample of m series, however, is chosen by a random process. Confining one's attention to the size (number of individuals, say) of the subsets (denote it by n_i for the ith subset) and the mean value x of some variate X (denote this by x_i for the ith subset), one can represent the sampling process by a drawing from a finite bivariate population (M marbles marked with n_i, x_i).

The author estimates the population mean of X, say $\bar{X}=\sum_{i=1}^M n_i x_i/\sum_{i=1}^M n_i$ by (1) $\bar{x}=(m\bar{\xi}-\bar{\bar{\xi}})/(m-1)$ or (2) $\bar{x}=\{m(M-1)\bar{\xi}-(M-m)\bar{\bar{\xi}}\}/\{M(m-1)\}$ for the cases of recurrent and nonrecurrent sampling, respectively. The quantities $\bar{\xi}=\sum_{\alpha=1}^m \nu_\alpha\xi_\alpha/\sum_{\alpha=1}^m \nu_\alpha$ and $\bar{\bar{\xi}}=m^{-1}\sum_{\alpha=1}^m \xi_\alpha$ are the weighted and unweighted averages of the observed values ξ_α of x. The statistics (1) and (2) are derived as (approximately) unbiased estimates by writing $y=\sum_1^m n_i x_i$, $z=\sum_1^m n_i$ (the n_i and x_i being regarded here as independent stochastic variables), expanding $E(y/z)$ in a Taylor series in the neighborhood of $\bar{y}$, $\bar{z}$, and neglecting moments of order higher than the second in the distribution of x and n.

The quantities (1′) $\mu_{\bar{x}}^2=\sigma_\xi^2/m$ and

$$(2') \qquad \mu_{\bar{x}}^2=(M-m)\sigma_\xi^2/(M-1)m$$

are similarly derived as estimates of the variance of the sample mean, where σ_ξ^2 is the observed variance of the ξ_α in a sample of size m. *A. A. Brown* (Alexandria, Va.).

TOPOLOGY

Colmez, Jean. Des espaces à écart. Revue Sci. **85**, 39–41 (1947).

There are given definitions of compactness, topological product, etc., involving a cardinal $\aleph$ which for $\aleph=\aleph_0$ reduce to the ordinary forms. In a sense, these generalized properties relate to spaces with monotone (basis for) uniform structures as do the ordinary ones to metric spaces. *R. Arens* (Los Angeles, Calif.).

Dieudonné, Jean. Sur un espace localement compact non métrisable. Anais Acad. Brasil. Ci. **19**, 67–69 (1947).

A locally compact nonnormal space is constructed which can be mapped in a one-to-one and continuous manner on a subset of the ordinary plane. This shows that nonmetric locally compact refinements of metrizable topologies are possible. *R. Arens* (Los Angeles, Calif.).

Sorgenfrey, R. H. On the topological product of paracompact spaces. Bull. Amer. Math. Soc. **53**, 631–632 (1947).

J. Dieudonné [J. Math. Pures Appl. (9) **23**, 65–76 (1944); these Rev. **7**, 134] asked whether $S\times S$ is "paracompact" if S is. The present article shows that $S\times S$ need not be normal when S is paracompact, thus settling the original question in the negative. *R. Arens* (Los Angeles, Calif.).

Dowker, C. H. Mapping theorems for non-compact spaces. Amer. J. Math. **69**, 200–242 (1947).

L'auteur généralise les théorèmes de Hopf et de Bruschlinsky sur les classes de représentations en sphères. Pour faire la réduction combinatoire du problème, il doit se servir d'une notion d'espace qui permet la transition aux complexes (infinis) devenue classique depuis les recherches d'Alexandroff. Les généralisations immédiates des recouvrements finis sont les recouvrements à étoiles finies (chaque élément du reçouvrement ne rencontre qu'un nombre fini d'autres) et les recouvrements localement finis (chaque point possède un voisinage ne rencontrant qu'un nombre fini d'éléments recouvrants). Ces trois espèces du recouvrement conduisent à la même notion de dimension dans les espaces normaux [ce n'est plus vrai pour des recouvrements arbitraires: voir J. Dieudonné, J. Math. Pures Appl. (9) **23**, 65–76 (1944); ces Rev. **7**, 134]. Par contre la théorie d'homologie et ses applications homotopiques dépendent essentiellement du choix du type de recouvrements admis; elles dépendent aussi de la manière dont on se sert des cocycles de Čech (des cocycles basés sur les recouvrements finis ou sur les recouvrements finis des sous-ensembles compacts ou sur des recouvrements infinis). Toutes ces notions conduisent à des généralisations différentes des théorèmes du Hopf et du Bruschlinsky. Les généralisations les plus satisfaisantes sont basées sur les recouvrements infinis, et elles sont possibles dans les espaces paracompacts [Dieudonné, loc. cit.], où chaque recouvrement possède un raffinement localement fini. C'est une classe très vaste qui contient les espaces compacts de Hausdorff et les espaces métriques séparables. Mais si l'on veut se borner à des recouvrements finis, la homotopie uniforme donne des résultats aussi élégants que ceux-là. La première groupe d'homologie de la droite devient alors le groupe additif des fonctions continues modulo le groupe des fonctions continues bornées. *H. Freudenthal* (Utrecht).

Reeb, Georges. Sur les variétés de niveau d'une fonction numérique. C. R. Acad. Sci. Paris **224**, 1324–1326 (1947).

Let f be a twice differentiable function, defined on the compact connected manifold V_n; suppose the singular points of f are isolated. Let $\bar{V}$ denote the union of the connected level surfaces of f, corresponding to singular values. Every component of the complement Ω of $\bar{V}$ is homeomorphic to

the product of a connected level surface of f and the open unit interval i; Ω is then homeomorphic to $V_{n-1}\times i$, where V_{n-1} is the union of a finite number of level surfaces. From the homology sequence of the pair $(V_n, \bar{V})$ one obtains relations between the Betti numbers of V_n, V_{n-1} and $\bar{V}$, and the ranks of certain subgroups of the homology groups; a consequence is the equation for the characteristics: $\chi(\bar{V})=\chi(V_n)+\chi(V_{n-1})$.

Similarly, by considering components of $\bar{V}$, and setting up a homology sequence, one gets relations involving the Betti numbers of V_{n-1} and $\bar{V}$ and the type numbers of the singular values. A consequence of these relations is the known fact that the alternating sum of the type numbers equals $\chi(V_n)$. *H. Samelson* (Ann Arbor, Mich.).

Reeb, Georges. Variétés feuilletées, feuilles voisines. C. R. Acad. Sci. Paris **224**, 1613–1614 (1947).

A manifold V_n is called leaved (feuilletée) if, roughly speaking, it is decomposed locally into (open) k-cells, the "elements," for a certain fixed k, $0<k<n$; an example for this is given by the elementary integrals of a completely integrable field of k-dimensional contact elements. Complete leaves (feuilles) are the components of V_n in the topology determined by the "elements" as base for open sets. Theorem 1. If V_n is compact, and if all leaves are regular or locally everywhere dense, then the closure of every regular leaf contains a compact leaf. Theorem 2. If V_q is a compact leaf with finite fundamental group π, then all leaves meeting a certain neighborhood of V_q are compact, and their fundamental groups are isomorphic with a subgroup of π; if also V_n is compact, and $q=n-1$, then all leaves are compact. Other related results are discussed. *H. Samelson* (Ann Arbor, Mich.).

Ehresmann, Charles. Sur les espaces fibrés différentiables. C. R. Acad. Sci. Paris **224**, 1611–1612 (1947).

The author introduces the concepts of differentiable fiber bundle and differentiable fiber map and states several propositions. Let E, F, B be differentiable manifolds; dim $B=n$. Proposition 1. If E is compact, then every differentiable map p, of rank n everywhere, of E onto B defines E as a differentiable fiber bundle over B. A field Φ of n-dimensional elements of contact in the differentiable fiber bundle E is called transversal (sécant), if at every point the element of the field and the tangent space to the fiber are linearly independent. Proposition 2. If Φ is a completely integrable transversal field of the differentiable fiber bundle E, with compact fiber F, then the complete integrals of Φ are coverings of the base space B under the projection; if B' is the smallest common covering of all the complete integrals then there exists a natural covering map of $B'\times F$ onto E, under which each $B'\times y$, $y\in F$, is mapped on a complete integral. Proposition 3. If Φ is a transversal field of the differentiable fiber bundle E, with compact fiber F, then every path from x to x' in the base space B determines a homeomorphism of F_x with $F_{x'}$; for closed paths one has automorphisms of F, which define the structure group of E. If Φ is completely integrable, then the automorphism depends only on the element of the fundamental group determined by the closed path. *H. Samelson* (Ann Arbor, Mich.).

Hu, Sze-tsen. Inverse homomorphisms of the homotopy sequence. Nederl. Akad. Wetensch., Proc. **50**, 279–287 = Indagationes Math. **9**, 169–177 (1947).

The homotopy groups of a space Y, of a subset Y_0 of Y, and the relative homotopy groups of Y modulo Y_0 are related by an exact homomorphism sequence (investigated in connection with fibre spaces by Hurewicz and Steenrod, by Eckmann and by J. H. C. Whitehead). It is proved that under reasonable geometric conditions some of the homomorphisms of the sequence are isomorphisms, which implies a direct sum decomposition of certain homotopy groups. *B. Eckmann* (Lausanne).

Young, Gail S. On compact fiberings of the plane. Bull. Amer. Math. Soc. **53**, 295–298 (1947).

If $f(E^2)=A$ is an interior transformation of the plane such that, for each two points x and y of A, $f^{-1}(x)$ and $f^{-1}(y)$ are homeomorphic, and such that each component of $f^{-1}(x)$ is compact, then no component of $f^{-1}(x)$ separates E^2 and A is a 2-manifold. If in addition $f^{-1}(x)$ is compact, then f is monotone and A is a plane. An immediate corollary is that there is no compact fibering of the plane other than a homeomorphism. If, in addition to the hypotheses of the first part of the theorem, each component of each set $f^{-1}(x)$ is locally connected, then it is shown that f is light. The author states that an extension of his proofs establishes that an interior transformation of a 2-manifold onto a 2-manifold such that each component of each point-inverse is an arc or a point is necessarily light. *W. W. S. Claytor* (Hampton, Va.).

Lifshitz, Jaime. A theorem on transformations of closed curves into themselves. Bol. Soc. Mat. Mexicana **3**, 21–25 (1946). (Spanish)

A proof is given of the following theorem. If J is a simple closed curve in the plane and T is a continuous transformation of J into J such that $T(P)\neq P$ for $P\in J$ (no fixed point), then as P goes around the curve J in the positive sense, the argument of the vector joining P to $T(P)$ increases by 2π. *J. H. Roberts* (Durham, N. C.).

Calugareanu, Georges. Sur les invariants topologiques attachés aux courbes et surfaces fermées. Disquisit. Math. Phys. **2**, 149–167 (1942).

The author discusses the conditions on F, under which a line integral of the form

$$I=\int_{t_0}^{t_1} F(x, y, z, x', y', z', x'', \cdots, x^{(n)}, y^{(n)}, z^{(n)})dt$$

represents a topological invariant, i.e., is invariant under arbitrary (differentiable) homeomorphisms of 3-space; it is hoped that this approach might yield new invariants for the knot problem. By approximating the homeomorphisms by polynomials and differentiating with respect to their coefficients, partial differential equations for F are obtained, which are formally identical with the Euler equations of calculus of variations. Special solutions are obtained by taking the total t-derivative of an arbitrary function Φ of $x', y', z', \cdots, x^{(n-1)}, y^{(n-1)}, z^{(n-1)}$. In order for I to be non-zero, Φ has to be multiple valued. An example in two dimensions is $\arctan y'/x'$; the corresponding I for a closed curve is the number of complete rotations of the tangent vector. Integral invariants of two curves in space and of a surface in space are treated in a similar way; the restricted group of homeomorphisms which leave the origin fixed is also considered. It is shown that the integrands of Gauss's linking integral, of the curvatura integra of a surface and of Kronecker's characteristic satisfy the appropriate equations of invariance. The paper concludes with some remarks on the possibility of transforming two congruences of curves into each other by a mapping of 3-space. *H. Samelson.*

GEOMETRY

Blumenthal, Leonard M. The metric characterization of φ-spherical spaces. Univ. Nac. Tucumán. Revista A. 5, 69–93 (1946). (Spanish)

This is another development of most of the results of an earlier paper by the author and G. R. Thurman [Amer. J. Math. 62, 835–854 (1940); these Rev. 2, 72]. In the present paper a set of k points ($k>1$) $p_1, \cdots, p_k$ is defined to be dependent if the determinant $|\phi(p_ip_j/\rho)|$ vanishes, where ρ is a positive constant, p_ip_j is the distance of the points p_i, p_j and of ϕ it is required merely that it is a real decreasing function with $\phi(0)=1$ and $\phi(d/\rho)=-1$, where d is the diameter of the space. In the former paper ϕ is cos (p_ip_j/d).

J. L. Dorroh (Kingsville, Tex.).

This review is to be replaced by the one in Errata, p. 709

Abellanas, Pedro. On the orientation in the bundle of rays in Euclidean or hyperbolic space. Revista Acad. Ci. Zaragoza (2) 1, 24–31 (1946). (Spanish)

The author discusses two methods of introducing an orientation in a bundle of rays. One method (local orientation) is based upon the consideration of the convex trihedral surface defined by three rays not in a plane and is equivalent to the ordinary orientation on a sphere. The other method (total orientation) takes into account all trihedral surfaces defined by a three-ray. It results that a bundle of rays has eight total orientations. *L. M. Blumenthal.*

Bloch, André, et Guillaumin, Gustave. Sur le volume des polyèdres non euclidiens. C. R. Acad. Sci. Paris 224, 1690–1692 (1947).

The volume V of a non-Euclidean tetrahedron satisfies the differential equation $dV=\pm\frac{1}{2}\sum l d\lambda$, where l is an edge and λ is the dihedral angle at that edge. This was proved in the elliptic case (upper sign) by Schläfli [Quart. J. Math. 2, 269–301 (1858)] and Richmond [ibid. 34, 175–177 (1903)]. Their proofs can easily be adapted to the hyperbolic case (lower sign). For a "quadrirectangular" tetrahedron (whose faces are four right triangles) in hyperbolic space, the following solution can soon be verified:

$$4V=L(\delta+\alpha)+L(\delta-\alpha)-L(\delta+\beta)-L(\delta-\beta)+L(\delta+\gamma)+L(\delta-\gamma)-2L(\delta),$$

where $\frac{1}{2}\pi-\alpha$, β, $\frac{1}{2}\pi-\gamma$ are the three acute dihedral angles, δ is given by

$$\cos\delta=\sin\alpha\sin\gamma/\sqrt{\sin^2\alpha-\sin^2\beta+\sin^2\gamma},$$

and $L(x)=-\int_0 \log\cos x\,dx$ ["N. J. Lobatschefskijs imaginäre Geometrie und Anwendung der imaginären Geometrie auf einige Integrale," Leipzig, 1904, p. 97; Coxeter, Quart. J. Math., Oxford Ser. 6, 13–29 (1935)]. This result remains valid in elliptic space if we make a suitable definition for $L(x)$ when x is complex. Since any polyhedron can be dissected into a finite number of quadrirectangular tetrahedra, it follows that the volume of any polyhedron is rationally expressible in terms of Lobatschefsky functions.

The authors reach the same general conclusion by direct reduction of the integral

$$\int\int\int dxdydz/[f(x,y,z,1)]^2,$$

where $f(x,y,z,t)=0$ is the absolute; only instead of $L(x)$ they use the "dilogarithm" dil $x=\int_1 \log(x-1)dx/x$, which satisfies

$$\text{dil}\,(1+e^{2ix})-\text{dil}\,(1+e^{-2ix})=-4i\int_0 x\tan x\,dx=4i[L(x)+x\log\cos x],$$

$$\text{dil}\,(1+e^{2ix})+\text{dil}\,(1+e^{-2ix})=-2x^2-\pi^2/6.$$

They remark that the content of a four-dimensional polytope is given by the same kind of expression, in agreement with another of Schläfli's results. *H. S. M. Coxeter.*

Puig Adam, P. Critical revision of the theory of equivalence of polygons. Revista Mat. Hisp.-Amer. (4) 7, 10–20 (1947). (Spanish)

In Hilbert's Grundlagen der Geometrie [7th ed., Berlin, 1930] a theory of area is developed for a plane geometry which is based on axioms I (1–3), II, III, IV (the continuity axioms of group V are not assumed). The theory depends on two notions of equivalence of simple polygons according to which (a) two simple polygons are equivalent by decomposition if they may be decomposed into the same finite number of pairwise congruent triangles, and (b) two simple polygons are equivalent by complementation provided pairs of polygons P_i, Q_i ($i=1,\cdots,n$) exist, equivalent by decomposition, such that the polygonal sums $P+P_1+\cdots+P_n$ and $Q+Q_1+\cdots+Q_n$ are equivalent by decomposition. In this paper these notions and the procedure based upon them are subjected to criticism. It is contended, for example, that though equivalence by decomposition and its transitive and additive properties are utilized as a basis for equivalence by complementation, no demonstrations of the properties are given, and it is asserted that transitivity of equivalence by complementation is more difficult to prove than statements in the Grundlagen suggest in view of the fact that polygons exist which cannot be juxtaposed to form a polygon. To overcome these objections the author proposes the following definition: two polygons P, Q are equivalent whenever Q (or a congruent image of it) may be obtained from P by adding algebraically to P a finite set of polygons, with those polygons which are added congruent, respectively, to those subtracted. This relation (which includes both kinds of decomposition) is easily shown to be reflexive, symmetric, transitive and additive. *L. M. Blumenthal.*

Abellanas, Pedro. On the postulates of order in the projective space of Steinitz-Rademacher. Revista Acad. Ci. Zaragoza (2) 1, 18–23 (1946). (Spanish)

The author considers the projective space defined by Steinitz and Rademacher [Vorlesungen über die Theorie der Polyeder, Springer, Berlin, 1934] and shows that the third axiom of order (if A, B separate C, D then A, B separate D, C and B, A separate C, D and D, C) is a consequence of the other order postulates and the axioms of incidence.

L. M. Blumenthal (Columbia, Mo.).

Walker, A. G. Finite projective geometry. Edinburgh Math. Notes no. 36, 12–17 (1947).

This is a descriptive article. It uses the cyclic representation of finite Desarguesian planes to illustrate constructions and problems of projective geometry.

M. Hall, Jr. (Columbus, Ohio).

Richmond, H. W. On isoptic families of curves. Edinburgh Math. Notes no. 36, 18–21 (1947).

Consider a pencil of lines, moving in a plane as a rigid figure. The author shows that, if two selected lines move so

as to remain tangent to two similar curves of specified type, every line of the pencil envelopes a curve of the same type. The theorem is valid when both the given curves are circles, or cycloids, or similar epicycloids or hypocycloids, or equiangular spirals or involutes of circles. There are infinitely many other curves having the same property, but except for the trivial case of a pencil rotating about a fixed point, no others are familiar curves. The locus of a point whence tangents to two curves make a fixed angle is their isoptic locus. The author does not determine this locus except for the familiar case of circles.

The method of treatment is noteworthy. The lines $x\cos\psi+y\sin\psi=p$ will envelop a curve if $p=f(\psi)$ is a given function of ψ. This function is easily written for any of the curves enumerated. Let two equations have the same $f(\psi)$ but with different constants; then the conditions of the theorem are satisfied if a linear combination of them is of the same type. Hence $f(\psi)$ must be the solution of a homogeneous linear differential equation with constant coefficients; it appears that the "D-operator" of this equation must have the factor D^2+1 and is otherwise arbitrary.

R. A. Johnson (Brooklyn, N. Y.).

Sergescu, P. Sur une proposition de Cayley. Bull. Sci. École Polytech. Timişoara 11, 22–29 (1943).

Cayley enunciated the following proposition, which is easily established. If $L_i: a_ix+b_iy+c_i=0$, $i=1, 2, \cdots, 6$, are lines in a plane, a condition that the intersection points of L_1 and L_2, of L_3 and L_4, and of L_5 and L_6 are collinear is that

$$\begin{vmatrix} a_1 & b_1 & c_1 & a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ a_4 & b_4 & c_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_5 & b_5 & c_5 \\ 0 & 0 & 0 & a_6 & b_6 & c_6 \end{vmatrix} = 0.$$

The author derives a generalization of this result in space of n dimensions, in the form of a determinant of order $n(n+1)$, and shows its equivalence to the n-rowed determinant of the coordinates of the intersections.

R. A. Johnson (Brooklyn, N. Y.).

Abramescu, N. Sur une classe de courbes généralisant les coniques. Bull. Sci. École Polytech. Timişoara 12, 142–145 (1946).

Thébault, V. Triangle et tétraèdre. Bull. Sci. École Polytech. Timişoara 11, 54–61 (1943).

Thébault, V. Sur le tranchet d'Archimède. Bull. Sci. École Polytech. Timişoara 11, 67–77 (1943).

Thébault, V. Géométrie du triangle et du tétraèdre. Bull. Sci. École Polytech. Timişoara 11, 220–262 (1944).

Thébault, V. Tétraèdres associés à deux tétraèdres donnés. Bull. Sci. École Polytech. Timişoara 12, 175–178 (1946).

Thébault, Victor. Nouvelles sphères associées au tétraèdre. C. R. Acad. Sci. Paris 224, 1614–1616 (1947).

Cavallaro, Vincenzo G. Notes sur la géométrie du triangle. Bull. Sci. École Polytech. Timişoara 9, 235–244 (1940).

Cavallaro, Vincenzo G. Sur la nouvelle géométrie du triangle. La configuration ayant pour éléments partiels la polaire trilinéaire du point de Lemoine et la distance brocardienne. Bull. Sci. École Polytech. Timişoara 10, 73–86 (1941).

Botez, Mihail St. Sur le problème de la projection bicentrale dans un espace à quatre dimensions. Bull. Sci. École Polytech. Timişoara 9, 257–272 (1940).

Botez, Mihail St. L'expression analytique de la perspective newtoniènne des polygones. Bull. Sci. École Polytech. Timişoara 10, 93–99 (1941).

Convex Domains, Extremal Problems

Efimoff, N. Étude des déformations infiniment petites de certaines classes de surfaces. Rec. Math. [Mat. Sbornik] N.S. 20(62), 27–53 (1947). (Russian. French summary)

Let S be the portion, between two parallel planes, of the surface of the smallest convex solid containing two ovals situated in these planes. Consider deformations of S of the form $x(u, v)+\epsilon z(u, v)+\epsilon^2 w(u, v)$, where x, z, w are vector-functions periodic in u in the strip $0\leqq v\leqq 1$, the first of which (x) is linear in v and defines S. Subject to the usual assumptions of differential geometry, the author proves that if these deformations (i) do not increase the distance of near points of the bounding ovals, and (ii) alter the lengths of smooth curves on S by at most $o(\epsilon^2)$, then they are of a very special type which he terms trivial. A similar result is established for the part of an oval surface of positive curvature bounded by a finite number of oval curves.

L. C. Young (Columbus, Ohio).

Haantjes, J. Symmetrization in the hyperbolic plane. Simon Stevin 25, 56–61 (1947). (Dutch)

According to Steiner, a closed curve is "symmetrized" by the process of shifting every section perpendicular to a given line till its midpoint lies on the line. This process diminishes the circumference L of any curve not symmetrical about the given line, while leaving the area O unchanged. Hence any closed curve other than a circle yields a shorter closed curve for some direction of the line. The author considers, in the hyperbolic plane, a closed curve whose geodesic curvature is everywhere greater than 1 and shows that this property is retained after symmetrization. He indicates a proof that, among such curves, the circle has the smallest circumference for its area. He states without proof the sharper result that for such a curve

$$L^2-O(O+4\pi)\geqq 4\pi^2\tanh^2\tfrac{1}{2}(r_u-r_i),$$

where r_i and r_u are the radii of the greatest inscribed and smallest circumscribed circles. *H. S. M. Coxeter.*

Moran, P. A. P. On a problem of S. Ulam. J. London Math. Soc. 21, 175–179 (1946).

The author proves in an elementary way that the area of the smallest convex cover (the convex hull) of a plane curve (not necessarily closed) of unit length is at most $(2\pi)^{-1}$; equality is valid for the semicircle of unit length.

W. Fenchel (Copenhagen).

Gustin, William. On the interior of the convex hull of a Euclidean set. Bull. Amer. Math. Soc. **53**, 299–301 (1947).

The following theorem is proved by induction. Any point interior to the convex hull of a set E in an n-dimensional Euclidean space is an interior point of the convex hull of some subset of E containing at most $2n$ points. (If "interior" is omitted, $2n$ may be replaced by $n+1$ in consequence of a well-known theorem.) The author seems not to be aware that this is contained in the results of Dines and McCoy [Trans. Roy. Soc. Canada. Sect. III. **27**, 37–70 (1933)] and Dines [Bull. Amer. Math. Soc. **42**, 353–365 (1936), in particular, p. 358] on systems of linear inequalities.

W. Fenchel (Copenhagen).

Algebraic Geometry

Todd, J. A. On algebraic curve branches. J. London Math. Soc. **21**, 233–240 (1946).

P. Du Val [Rev. Fac. Sci. Univ. Istanbul. (A) **7**, 107–112 (1942); these Rev. **6**, 17] has given a method of characterizing the structure of an algebraic branch lying in an r-dimensional space in terms of a finite set of integers called "characters." Todd extends this result by giving necessary and sufficient conditions that a set of integers be the characters of a branch. The main result is the following. Let $i_1, i_2, \cdots$ be a sequence of positive integers such that (1) for each k there is a λ_k such that (*) $i_k = i_{k+1} + i_{k+2} + \cdots + i_{k+\lambda_k}$; (2) $i_n = 1$ for some n. The "restriction" of a term i_k of the sequence is the number of equations (*) in which i_k appears on the right-hand side. If r is any integer greater than or equal to the maximum of the restrictions of the terms of the sequence then there exists an algebraic branch, lying in a space of r dimensions, whose consecutive neighboring singularities have multiplicities $i_1, i_2, \cdots$. *R. J. Walker.*

Amodeo, Federico. Il problema della gonalità. Rend. Accad. Sci. Fis. Mat. Napoli (4) **11**, 41–72 (1941).

A plane curve of gonality k is one that has a characteristic series g_k^1 and no simply infinite linear series of lower degree. The author determines the gonality of plane curves of given order and with given singularities, classifying singular curves of order m from those of maximum gonality $m-1$ (nonsingular regular curves) to those of gonality 1 (curves having an $(m-1)$-fold point). He obtains the envelope of the lines that cut out the g_k^1 in each case. He also continues the work of a previous paper on normal plane curves of any given gonality k, classifying them in $k-1$ types.

T. R. Hollcroft (Aurora, N. Y.).

Bigi, B. La "piccola variazione" di una curva algebrica reale connessa, con speciale riguardo al caso di Harnack. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 27–30 (1947).

Harnack [Math. Ann. **10**, 189–198 (1876)] proved that a real algebraic curve of genus p has at most $p+1$ circuits. His method, that of "small variations," has been used by many writers. A mathematical interpretation will be found in Anhang F of F. Severi's Vorlesungen über algebraische Geometrie [Teubner, Leipzig, 1921]. In this note the author states necessary and sufficient conditions that it should be possible to deduce by "small variations" from a real connected algebraic curve which consists of k real irreducible components without multiple points a curve whose real part has the maximum number of circuits compatible with the genus. *D. Pedoe* (London).

Bigi, B. La "piccola variazione" di una curva algebrica reale connessa, con speciale riguardo al caso di Harnack. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 125–129 (1947).

Proof of the theorem enunciated in the note reviewed above. *D. Pedoe* (London).

Arvesen, Ole Peder. Courbes de Chasles supérieures. Norske Vid. Selsk. Forh., Trondhjem **14**, no. 33, 123–126 (1941).

Let the tangential equation of a curve in homogeneous coordinates be

$$w^{m-r}\varphi_r(u, v) + w^{m-r-1}\varphi_{r+1}(u, v) + \cdots + \varphi_m(u, v) = 0.$$

Then the curve $s_n: (m-r)w^n = \sum_{i=1}^{m-r} w_i^n$ for n a positive integer is called a higher Chasles curve. The curve s_n admits a linear homogeneous substitution, for example, a rotation around the origin. To an arbitrary algebraic curve there exist several s_n (the case $n=1$ excepted), but by certain conventions it is possible to distinguish a unique s_n. The properties of the s_2 are studied especially. *E. Bodewig.*

Arvesen, Ole Peder. Quelques applications des surfaces s_n. Norske Vid. Selsk. Forh., Trondhjem **15**, no. 29, 111–114 (1942).

The equations of a conic in space of which two conjugate diameters are given and of a quadric of which three conjugate diameters are given. *E. Bodewig* (The Hague).

Arvesen, Ole Peder. Remarques sur les courbes Γ. Norske Vid. Selsk. Forh., Trondhjem **15** (1942), no. 40, 153–156 (1943).

In contradiction to his former terminology the author assumes that the curve of Chasles of a curve Γ is irreducible. For this, $\Gamma \equiv \sum_{i=r}^{m} w^{m-i}$. The curve $\varphi_i(u, v) = 0$ must have only one tangent at infinity. The generating curve of Γ has the class $(m-r)(r+1)$. Only two curves of this type are known: Rolle's curve and the leaf-curve. *E. Bodewig.*

Arvesen, Ole Peder. Sur les courbes s_n négatives. Norske Vid. Selsk. Forh., Trondhjem **17** (1944), no. 29, 114–117 (1945).

The author studies the properties of the curve s_{-2} in the case when the original curve is a pair of points.

E. Bodewig (The Hague).

Niculescu, Alexandru V. On spherical Ţiţeica curves. Bull. Sci. École Polytech. Timişoara **12**, 64–68 (1945). (Romanian)

Drăgilă, Pavel. Détermination de quelques courbes gauches. Bull. Sci. École Polytech. Timişoara **12**, 179–182 (1946).

Ničе, W. Die Raumkurve der Tangentialpunkte eines Ebenenbüschels und eines Flächenbüschels zweiter Ordnung. Bull. Intern. Acad. Croate. Cl. Sci. Math. Nat. **34**, 25–26 (1941).

This is the first of six contributions on algebraic curves and surfaces. The planes P_t through a fixed line p in [3] space touch the quadric surfaces of a pencil φ^2 at points which lie on a twisted quintic curve T^5. Any point y has a

polar line q with regard to this pencil (q being common to all polar planes of y). As y describes the line p, q generates a quadric surface E_p. The points of contact of all tangent lines from y to the quadrics of the pencil lie upon ∞^1 of conics which generate a cubic surface. The quintic curve T^5 is on both surfaces, E_p and the cubic, which have a line also as residual intersection. Trisecants of T^5 are generators of a quadric surface. *H. W. Turnbull* (St. Andrews).

Niče, W. Die Flächen vierter Ordnung der Tangentialpunkte eines Ebenenbüschels und eines F^2 Bündels der Flächen zweiter Ordnung. Bull. Intern. Acad. Croate. Cl. Sci. Math. Nat. **34**, 26–28 (1941).

[Cf. the preceding review.] Given a straight line p and a net (∞^2) of quadric surfaces F^2, and the planes through p, then the locus of the points of contact of the planes with the quadrics is a quartic surface Δ. Properties of Δ are studied: it contains p and the 8 associated points common to all the quadrics; Δ is an interesting example of a quartic which possesses a line but has no double point. Each line p has its quartic surface Δ, and all such surfaces contain C^6, the Jacobian curve of the net (the locus of the vertices of the cones belonging to the net). Special cases of Δ are considered, when p is no longer in general position.

H. W. Turnbull (St. Andrews).

Niče, Villim. Der Bündel von Flächen zweiter Ordnung. Bull. Intern. Acad. Croate. Cl. Sci. Math. Nat. **35**, 42–43 (1945).

[Cf. the two preceding reviews.] If p goes through two of the eight associated points, its quartic surface Δ has these for triple points and contains the line p also. Any plane through p cuts the Jacobian curve in six points (including these two associated points) which lie on a conic. Of the results stated as hitherto unknown, one was given by Hesse [J. Reine Angew. Math. **49**, 279–332 (1855); 298] and the other by Edge [Proc. London Math. Soc. (2) **43**, 302–315 (1937); 305]. *H. W. Turnbull* (St. Andrews).

Niče, Vilim. Beitrag zur konstruktiven Behandlung der Regelflächen dritter Ordnung. Bull. Intern. Acad. Croate. Cl. Sci. Math. Nat. **35**, 89–91 (1945).

[Cf. the three preceding reviews.] This deals with the construction and the properties of ruled cubic surfaces [3] by a general procedure which covers the two main types of such surfaces (those with real and those with imaginary pairs of cuspidal points upon the double directrix) and which extends to the third or Cayleyan type of surface. Every generator, in the first two types, meets one simple and one double directrix line. The pencil of planes through a given generator cut the surface in ∞^1 conics. The point M where this generator cuts the simple directrix will have a polar line for each conic; the ∞^1 of polars generate a polar quadric surface. This surface is the key to the sequel.

H. W. Turnbull (St. Andrews).

Niče, Vilim. Ein Beitrag zur Erforschung der gemeinsamen Eigenschaften ebener Kurven 3. und 4. Ordnung vom Geschlecht Null. Bull. Intern. Acad. Croate. Cl. Sci. Math. Nat. **35**, 98–99 (1945).

[Cf. the four preceding reviews.] This is a discussion of rational cubic and quartic curves in a plane, based on two fixed conics C and S and a variable conic whose envelope is the cubic or quartic curve. The treatment is reminiscent of Maclaurin's, both for its use of pedal curves and its organic description of the cubic or quartic. *H. W. Turnbull.*

Niče, Vilim. Kurven und Flächen 3. und 4. Ordnung entstanden mittels der quadratischen Inversion. Bull. Intern. Acad. Croate. Cl. Sci. Math. Nat. **35**, 140–151 (1945).

[Cf. the five preceding reviews.] Take the harmonic conjugate P' of the point P with regard to a fixed conic, such that the line PP' passes through a fixed point O. If P describes a conic, P' describes a rational quartic: if the conic passes through O, the quartic becomes a rational cubic curve. Properties are developed.

Next let the figure be in [3], with quadric surfaces replacing the conics. The locus of P' is a cubic or quartic surface, if that of P is a quadric. Both cubic and quartic surfaces are discussed with particular reference to nodes, real points and lines, and both conical and circular sections. The quartic surface has one or more nodes and one trope: cases are classified according to the positions of the given quadrics. For the case of one node and the one trope, twelve lines lie on the quartic, four through the node and meeting the trope, and eight more intersecting in pairs on the trope. At most 27 planes cut the surface in a conic, and two straight lines, or else in the conic reckoned twice. The cyclic case is also considered, when the base quadrics include spheres.

H. W. Turnbull (St. Andrews).

Niče, Vilim. The locus of the points of tangency of a pencil of planes with a pencil of quadric surfaces. Rad Hrvatske Akademije Znanosti i Umjetnosti. Razred Mat.-Prirodoslov. **84**, 65–68 (1941). (Croatian)

Niče, Vilim. Quartic surfaces as the locus of points of tangency of a pencil of planes with a bundle of quadric surfaces. Rad Hrvatske Akademije Znanosti i Umjetnosti. Razred Mat.-Prirodoslov. **84**, 69–76 (1941). (Croatian)

Niče, Vilim. On bundles of quadric surfaces. Rad Hrvatske Akademije Znanosti i Umjetnosti. Razred Mat.-Prirodoslov. **85**, 163–169 (1942). (Croatian)

Niče, Vilim. Contribution to the constructive treatment of the ruled cubic surfaces. Rad Hrvatske Akademije Znanosti i Umjetnosti. Razred Mat.-Prirodoslov. **85**, 286–298 (1942). (Croatian)

Niče, V. A contribution to the study of the common properties of curves of the third and fourth order of genus zero. Rad Hrvatske Akademije Znanosti i Umjetnosti. Razred Mat.-Prirodoslov. **86**, 55–61 (1945). (Croatian)

Niče, Vilim. Curves and surfaces of third and fourth order obtained by quadratic inversion. Rad Hrvatske Akademije Znanosti i Umjetnosti. Razred Mat.-Prirodoslov. **86**, 153–194 (1945). (Croatian)

Summaries of these six papers are reviewed above.

Sullivan, C. T. Some investigations in the projective differential theory of scrolls. Trans. Roy. Soc. Canada. Sect. III. (3) **40**, 1–15 (1946).

The author considers scrolls (ruled surfaces in three dimensions) whose asymptotic curves are cubics, and then more general scrolls. He bases the work upon the ordinary simultaneous second order linear differential equations for y and z in terms of x, where (x, y, z) are the Cartesian coordinates of a point in space, and where the differential geometry is developed. More purely geometrical methods and consequences are reserved for a sequel. Among results here proved are (1) that asymptotic curves of a scroll which belong to linear complexes are projectively equivalent, and

(2) that the correspondence between the curves and complexes may be either (1, 1) or (2, 1) when the scroll itself belongs to a nonspecial linear congruence. This alternative, which is at variance with a long standing result, is illustrated by examples. *H. W. Turnbull* (St. Andrews).

Godeaux, Lucien. Sur un faisceau de surfaces du sixième ordre. Simon Stevin 25, 49–55 (1947).

A pencil of sextic surfaces in [3] is considered, whose plane sections are pencils of Halphen sextic curves. Such a pencil of surfaces is found to depend upon F, a sextic surface possessing a double curve D of order nine, together with a cubic surface Φ reckoned twice. The curve D, which is the intersection of F and Φ, has four triple points M_i, which also are double points upon the cubic Φ and triple points upon the sextic F. Also the surfaces F and Φ touch each other at all points of the curve D.

Referred to the four points M_i as frame of reference the equation of the cubic is $\sum X_i^{-1}=0$, and that of the sextic surface is

$$\sum_{1,2,3}\mu_1^2(X_2X_3+X_3X_4+X_4X_2)^3=2X_4^3(\mu_2\mu_3X_1^3+\mu_3\mu_1X_2^3+\mu_1\mu_2X_3^3).$$

Suppressing the exponents 3 we have a quadratic equation which is that of a quadric cone inscribed in the cubic surface. The ∞^2 of cones (μ_i being parameters) are associated with an ∞^3 of twisted cubic curves on Φ. Geometrical and algebraic methods are used, the cubic surface being transformed to a plane by means of the relations

$$\rho X_1=(x_2+x_3)(x_3-x_1)(x_1-x_2),$$

etc., with signs $+--, -+-, --+, +++$, to give X_1, X_2, X_3, X_4, respectively. The equations

$$\sum_{1,2,3}\mu_1(x_2-x_3)^3(x_3+x_1)^3(x_1+x_2)^3=0,$$

and the same with the exponents 3 suppressed, are the plane equivalents of the nonic curve D and the twisted cubics. *H. W. Turnbull* (St. Andrews).

Longhi, Ambrogio. Sulla geometria degli S_k di un S_r. Comment. Math. Helv. 19, 320–346 (1947).

B. Segre introduced the concept of associated linear spaces in S_r. The author generalizes the concept as follows. In S_r, ν linear spaces S_k are associated and of species μ, $1\leq\mu<\nu-1$, $\mu>\nu-\binom{k+1}{k+1}$, when every linear complex of S_k which contains at least $\nu-\mu$ spaces S_k contains as a consequence the μ remaining spaces S_k. He studies the characteristics of various sets of associated S_k, including rational involutions of sets belonging to an algebraic system of spaces S_k and varieties generated by certain sets. *T. R. Hollcroft.*

Differential Geometry

Biran, Lutfi. Mouvement à un paramètre. Rev. Fac. Sci. Univ. Istanbul (A) 12, 208–229 (1947). (French. Turkish summary)

The author studies the motion of a rigid body, using the dual vector calculus [cf. Blaschke, Vorlesungen über Differentialgeometrie, v. 1, Springer, Berlin, 1930]. A dual velocity vector is defined, which combines the angular velocity and the velocity of a reference point in the body. It is shown how this vector as function of the time determines the motion. Darboux's treatment of the rotation of a body around a fixed point is carried over to the general case by considering a rotation of the dual unit sphere in itself; one is led to a Riccati equation, formally the same which Darboux found, only now with dual coefficients. Applications to the composition of motions, to ruled surfaces and related topics are given. *H. Samelson.*

De Cicco, John. The affinilong near-Laguerre transformations. Univ. Nac. Tucumán. Revista A. 5, 301–319 (1946).

An "affinilong line transformation" of the plane carries at most ∞^2 circles into circles. If it converts exactly ∞^2 circles into circles it is termed "an affinilong near-Laguerre transformation." The author determines completely the set of all affinilong near-Laguerre transformations. The differential equation in equilong coordinates $x, y, p=dy/dx$ of a circle being (1) $p''=0$, for any circle which becomes a circle under an affinilong transformation $X=\varphi(x)$, $Y=\psi(x, y)$ the differential equation (1) must be preserved, which leads to the differential equation (with four constants $a:b:c:d:t$) (2) $p'(2ty+ax^2+2bx+c)=tp^3+2(ax+b)p-2(ay-d)$ for our circles. By means of the group of Laguerre transformations systems (2) may be classified in four distinct types A, B, C, D. The author shows that any affinilong near-Laguerre transformation is of the form L_2TL_1, where L_1 and L_2 are Laguerre transformations and T is any of the affinilong correspondences which carry any of the types A, B, C into a type A or B or C. [In the differential equation (11) read $-\varphi_x^2\psi_{xxy}p^3$ instead of $-\varphi_x^2\psi_{xy}p^3$.] *V. Hlavatý* (Prague).

Kasner, Edward, and De Cicco, John. Projective differential invariants of a cusp. Univ. Nac. Tucumán. Revista A. 5, 289–299 (1946).

If a plane curve has a cusp at the origin, it may be represented by an equation of the form (1) $y=\sum_{n=3}^{\infty}c_nx^{\frac{1}{n}}$, $c_3\neq0$. It is shown that under the group of collineations preserving the form of equation (1) the relative differential invariant of lowest order is one of order seven and that the next one is of order eight. Explicit formulas for these invariants in terms of the coefficients c_n are given. *J. E. Wilkins, Jr.*

Chang, Su-Cheng. Contributions to projective theory of singular points of space curves. Trans. Amer. Math. Soc. 61, 369–377 (1947).

It is shown with the help of configurations defined in the neighborhood of order eight that the projective differential geometry of a space curve in a neighborhood of an inflexion point can be studied by means of the canonical expansions

$$y=Ax^3-3A^2x^5+Vx^7-2A^3x^8+\cdots,$$
$$z=Ax^4-3A^2x^6+Vx^8+\cdots,$$

in which the invariants A and V may be characterized by means of certain simple cross ratios. *J. E. Wilkins, Jr.*

Hsiung, Chuan-Chih. Plane sections of the tangent surfaces of two space curves. Duke Math. J. 14, 151–158 (1947).

If C and C' are curves in ordinary space intersecting but not tangent at a point P at which they have the same osculating plane, the author in a previous paper [same J. 10, 539–546 (1943); these Rev. 5, 76] has studied the points Q and Q' in which a general plane π intersects the tangents of C and C'. Let Γ and Γ' be the sections of the tangent surfaces of C and C' made by π. Then Γ and Γ' have a common tangent at Q and Q'. In this paper a study is made of certain osculants of Γ and Γ'. The general results are also

specialized to the case in which C and C' are the asymptotic curves of a nonruled surface. *J. E. Wilkins, Jr.*

Maeda, Kazuhiko. On an infinity of cylindroids associated with a tangent to a surface. Tôhoku Math. J. **49**, 302–304 (1943).

The following result is proved and slightly generalized. Let t be a nonasymptotic tangent at a regular point O on a surface. We intersect this surface with a variable plane through t and construct the straight line through the second and third centers of curvature at O of this intersection. Then this straight line generates a ruled cubic through t. This cubic intersects each plane perpendicular to t in a pair of straight lines that meet t. *P. Scherk.*

Arghiriade, E. Sur les points multiples de la ligne parabolique d'une surface. Bull. Sci. École Polytech. Timişoara **12**, 59–63 (1945).

Halphen [Bull. Soc. Math. France **3**, 28–37 (1874); Oeuvres, v. 1, pp. 317–328] has shown that if a surface S at a point M has contact of the third order with a quadric, then M is a double line of the flecnodal curve. It is one of the purposes of this paper to prove the following theorem: if a surface has third order contact at a parabolic point with a quadric cone, that point is a double point of the parabolic curve. More generally it is demonstrated that, if the surface has nth order contact with a quadric cone, the point is a multiple point of order $n-1$ of the parabolic curve. The converse theorem is not true if $n>2$; in fact, if $n>2$, the order of contact with a quadric cone need only be three. *V. G. Grove* (East Lansing, Mich.).

Pantazi, Al. Sur le problème de Koenigs. Disquisit. Math. Phys. **1**, 357–368 (1941).

Le problème de Koenigs consiste en la recherche des couples de courbes gauches en correspondance ponctuelle biunivoque telle que les plans osculateurs en un couple quelconque de points homologues se coupent suivant la droite joignant les deux points. Deux courbes ainsi associées sont dites [Bianchi] transformées asymptotiques l'une de l'autre. L'auteur étudie le problème de la recherche des courbes Γ simultanément transformées asymptotiques de deux courbes données C_0 et C_1. Il commence par définir une correspondance (fondamentale) F entre deux courbes gauches C_0 et C_1 par la propriété qu'étant donnés deux points homologues A_0 et A_1 de C_0 et C_1 il existe un complexe linéaire osculateur à C_0 en A_0 et à C_1 en A_1. Il montre ensuite que les transformées asymptotiques communes Γ de deux courbes gauches C_0 et C_1 sont sur la surface réglée R engendrée par l'intersection (A_2A_2) des plans osculateurs en deux points A_0 et A_1 de C_0 et C_1 se correspondant dans une correspondance F. Les courbes Γ transformées asymptotiques de C_0 et C_1 forment sur R une famille ∞^1 définie par une équation de Riccati. Sur deux courbes Γ quelconques (Γ_0 et Γ_1) les génératrices de R déterminent une correspondance F, d'où l'existence de ∞^1 courbes Γ' situées sur la surface réglée R' lieu de A_0A_1 (chaque courbe Γ' étant transformée asymptotique de Γ_0 et de Γ_1), la configuration formée par les deux familles (Γ), (Γ') donnant lieu à un théorème de permutabilité analogue à ceux auxquels conduisent les transformations par congruences W ou les couples de congruences stratifiables. En ce qui concerne les surfaces R, R', il est montré qu'elles admettent une quadrique de raccordement le long de deux génératrices homologues, et que les courbes Γ et Γ' sont les enveloppes, sur R et R', des rayons de la congruence de nappes focales R, R'. L'auteur définit ensuite les couples de surfaces réglées R et R' stratifiables, ramène leur recherche à celle des familles de quadriques à un paramètre ayant des quadrilatères gauches pour courbes caractéristiques, et signale les liens qui existent entre de telles familles de quadriques, les courbes en correspondance F, les transformations asymptotiques d'une même courbe ou les transformations par congruences W des surfaces réglées. *P. Vincensini* (Besançon).

Vincensini, Paul. Recherches sur la déformation des surfaces. Ann. Sci. École Norm. Sup. (3) **63** (1946), 255–288 (1947).

The purpose of this paper is to discuss the behavior of a congruence C associated with a surface Σ when Σ is subjected to an arbitrary deformation. If the central points of the lines of C are subjected to the condition of lying in the corresponding tangent planes of Σ it is found that the lines of C may be constructed in the following manner. Choose on Σ an arbitrary one parameter family of geodesics (v=constant); draw on the tangent to the orthogonal trajectory of these geodesics through the point M of Σ a line segment MI of length $V\sqrt{G}$ (the element of arc on Σ being $ds^2=du^2+Gdv^2$), V being an arbitrary function of v alone; through the point I so constructed draw a line δ parallel to the tangent to the geodesic through M of the given family. The line δ generates the desired congruence C. Among the properties of such congruences are the following. (a) The developables of C correspond to a conjugate net on Σ and every surface applicable to Σ. (b) On Σ there is a nonconjugate net invariant under all deformations of Σ; the tangents to the curves on Σ corresponding to the developables of C separate the tangents to this invariant net harmonically. Relations between the focal points of the lines of such congruences C and surfaces of revolution are found, a typical theorem being: if a congruence C whose lines lie in the tangent planes to a surface Σ is deformed by an arbitrary deformation of Σ in such a manner that the pairs of focal points on the lines of C are equidistant from the point of contact on Σ, the central points of the lines of C being constrained to lie in the tangent plane of Σ, then Σ is applicable to a surface of revolution. Finally a study is made of congruences C with the property of having one of the curves corresponding to its developables invariant under a deformation of the surface Σ. A result may be stated as follows: the invariance of one of the focal points on each line of C when Σ is deformed arbitrarily imposes the invariance of one of the two families of developables and conversely. *V. G. Grove* (East Lansing, Mich.).

Wang, Hsien-Chung. The projective deformation of nonholonomic surfaces. Duke Math. J. **14**, 159–166 (1947).

The author defines a projective deformation as a correspondence of the elements of two nonholonomic surfaces S and $\bar{S}$ such that to each pair of corresponding elements E and $\bar{E}$ of S and $\bar{S}$, respectively, there exists a projective transformation carrying E to $\bar{E}$ and carrying the neighbourhood of the first order of E to that of $\bar{E}$. He defines ordinary frames and associated projectivities and obtains a necessary and sufficient condition for the projective deformability of S and $\bar{S}$ in terms of a correspondence of ordinary frames at points of S and $\bar{S}$. The associated projectivities are preserved in a projective deformation. The author proves conversely that a mapping of two nonholonomic surfaces such that tangent directions correspond and correlated directions

with respect to the associated projectivity are preserved is a projective deformation. *E. T. Davies* (Southampton).

Gheorghiu, Gh. Th. Sur une classe de surfaces. Bull. Sci. École Polytech. Timişoara **10**, 87–92 (1941).

If K is the curvature of a surface at point P and d is the distance from the origin O to the tangent plane at P, then K/d^4 is a centro-affine invariant with O the center. When this invariant is constant over the surface the affine normals are concurrent. This case was first studied by Tzitzeica who suggested as a generalization surfaces for which the invariant is constant along one of the two families of asymptotic curves. The present paper is a first effort at studying this generalization. The author introduces the above invariant condition into the fundamental centro-affine partial differential equations of a surface and attempts to integrate them. One broad class of solutions consisting of ruled surfaces is reduced to the solution of an ordinary linear differential equation of the third order. Another special class of solutions is obtained explicitly to furnish examples which are nonruled. *J. L. Vanderslice* (College Park, Md.).

Rozet, O. Sur certaines formes différentielles d'une surface. Bull. Soc. Roy. Sci. Liége **15**, 117–119 (1946).

Rozet, O. Sur certaines formes différentielles d'une surface. II. Bull. Soc. Roy. Sci. Liége **15**, 188–191 (1946).

In addition to the usual first and second fundamental forms, the author considers the cubic form

$$\varphi_3 \equiv b_{\alpha\beta,\gamma}dx^\alpha dx^\beta dx^\gamma,$$

where $b_{\alpha\beta}$ is the second fundamental tensor. The solution curves of $\varphi_3=0$ are the lines of Gournerie, details of which are given for a general perisphere, the Dupin cycloid, ruled surfaces and quadrics. For a nondevelopable surface it is shown that the Riemannian curvature of the second fundamental form is equal to $H/2-I$, where H is the mean curvature and I is an invariant which in tensor notation (not that of the author) equals $\frac{1}{4}b^{-1}\{b_{\sigma 1,1}b_{\rho 2,2}-b_{\sigma 2,1}b_{\rho 2,1}\}b^{\sigma\rho}$, where b is the determinant of $b_{\alpha\beta}$ and $b^{\sigma\rho}$ are defined by the equations $b^{\sigma\rho}b_{\rho\tau}=\delta_\tau^\sigma$. An equivalent expression is given by É. Cartan [Bull. Sci. Math. (2) **67**, 8–32 (1943), p. 18; these Rev. **7**, 30]. *C. B. Allendoerfer* (Haverford, Pa.).

Vagner, V. On the problem of determining the invariant characteristics of Liouville surfaces. Abh. Sem. Vektor- und Tensoranalysis [Trudy Sem. Vektor. Tenzor. Analizu] **5**, 246–249 (1941). (Russian) [MF 15613]

After some calculations leading to a reformulation of the problem and a discussion of the number of independent integrals it is stated that the invariant characterization in question is very complicated and requires for its explicit expression lengthy computations. *G. Y. Rainich.*

Su, Buchin. On the surfaces whose Wilczynski quadrics all touch a fixed plane. Univ. Nac. Tucumán. Revista A. **5**, 363–373 (1946).

The author considers the problem of determining the surfaces whose Wilczynski quadrics are tangent to a fixed plane. The results obtained may be summarized as follows. If the two asymptotic tangents at a generic point of a nonruled surface intersect a fixed plane at two points Z_1 and Z_2, each one of the plane nets (Z_1) and (Z_2) must be the Laplace transform of the other. A necessary and sufficient condition that the quadrics of Wilczynski all touch the fixed plane is that this transformation of Laplace is a projective applicability. In this case the projective linear element of each net is equal to that of the surface and the point Z_3 of contact of the fixed plane with the quadric of Wilczynski is the intersection of the two tangents of the nets (Z_1) and (Z_2) which are distinct from the line joining Z_1 and Z_2. The Wilczynski quadrics of a surface are paraboloids if and only if the affine quadratic fundamental form of the surface has null curvature. *P. O. Bell* (Lawrence, Kan.).

Su, Buchin. Descriptive collineations in spaces of K-spreads. Trans. Amer. Math. Soc. **61**, 495–507 (1947).

Among the various geometries of K-spreads treated by Douglas [Math. Ann. **105**, 707–733 (1931)] the affine and the descriptive geometries are the generalizations of the affine and the projective geometries of paths. Affine and projective collineations of spaces of paths have been studied by Knebelman [Amer. J. Math. **51**, 527–564 (1929)]. In this paper the author applies the method of Lie derivation to generalize the results of Knebelman on projective collineations. The conditions of integrability of the equations determining a descriptive collineation are given in terms of the Lie derivative of the descriptive derivative of the descriptive curvature tensor. In the final paragraph it is proved that if a space of K-spreads admits r linearly independent infinitesimal descriptive collineations it admits the r-parameter group generated by them. *E. T. Davies.*

***Cartan, É. Leçons sur la Géométrie des Espaces de Riemann.** 2d ed. Gauthier-Villars, Paris, 1946. viii +378 pp. 650 francs.

The book is pervaded with tensor analysis from first to last, and yet tensor formalism for its own sake is rather underplayed. In this sense, this is not a book from which to learn the skill of tensor formalism; just as a book in complex variables, if leaning in a geometric direction, need not be the appropriate source from which to learn the technique of power series manipulation. The most engaging feature of the book, fully retained in the new edition, is the insistence on problems in the large, or of global incipience at least. There are ample discussions of covering spaces and fundamental groups for complete Riemannian spaces of constant curvature, positive, zero or negative. In notes III and IV it is proved that for any complete space whose (nonconstant) curvature is everywhere nonnegative, the universal covering space is of the type of an ordinary neighborhood and can be covered by one system of normal coordinates.

Entirely new is first of all a chapter on symmetric spaces. In tensor terms, these are spaces for which the Riemann-Christoffel tensor has covariant derivative zero, $R_{ij\beta l,m}=0$. Geometrically they are characterized by the property that a symmetry with respect to any point is an isometry, a symmetry being a transformation $x_i'=-x_i$ in terms of normal coordinates. Virtually all occurring spaces or domains, in particular, in several complex variables, with multiply transitive groups of motions on them, are of this kind and it is this "symmetry" rather than the transitive group of motions which is responsible for the attendant non-Euclidean geometry in a generalized sense. In the present book the author has unfortunately written but one brief chapter about them; but it is elegant and illuminating and fully supplementary to the author's previous publications on the topic.

New also is a full-length chapter on Lie groups of motions, replete with contents. Among others the following "special" result is obtained. If a four-dimensional group acts on a three-dimensional complete, simply-connected space, then the space is either Euclidean, or the product of a two-sphere with a line, or a three-sphere. A somewhat formal procedure is also given for deciding which Lie algebras leave a line element invariant, and for a given Lie algebra which line elements are invariant under it.

Finally there is a new chapter which methodically unifies the problem of deciding the (local) equivalence of two metrics with Killing's approach to determining the largest group of motions for any one metric; in both cases the decision is rendered by at most $n(n+1)/2$ successive covariant derivations of the Riemann-Christoffel tensor and no more. *S. Bochner* (Princeton, N. J.).

de Rham, Georges. Sur la théorie des formes différentielles harmoniques. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **22**, 135–152 (1946).

This is a very concise exposition (leaving many details to be filled in by the reader) of a new method of treatment of the theory of harmonic forms [cf. W. V. D. Hodge, The Theory and Applications of Harmonic Integrals, Cambridge University Press, 1941; P. Bidal et G. de Rham, Comment. Math. Helv. **19**, 1–49 (1946); these Rev. **2**, 296; **8**, 93], yielding essentially new results as well as simpler proofs for the known theorems. Let V be a compact orientable Riemannian variety of dimension n, d the differentiation operator (on differential forms on V), and $*$ the well-known operator, associating with each form α of degree p its adjoint $*\alpha=\alpha^*$ of degree $n-p$; put $(\alpha,\beta)=\int_V \alpha\cdot(*\beta)$ for any two forms α,β of the same degree, so that $(\alpha,\alpha)>0$ for $\alpha\neq 0$; the operators δ,Δ are defined by $\delta\alpha=(-1)^{np+n+1}*d*\alpha$ (for α of degree p), $\Delta=d\delta+\delta d$; harmonic forms are defined as those satisfying $\Delta\alpha=0$ (which, for α regular over the whole of V, implies $d\alpha=\delta\alpha=0$, i.e., harmonicity in Hodge's sense). Using Hodge's parametrix, the author gives a new and short proof for the fundamental existence theorem in the form already given by him [Bidal and de Rham, loc. cit.]: necessary and sufficient, for the existence of a solution μ for $\Delta\mu=\beta$, is that β is orthogonal (in the sense of the scalar product (α,β)) to all harmonic forms of its degree. From this the existence of a form $g_p(x,y)$ for each degree ("Green's form") is deduced, with the following properties: g_p is defined on $V\times V$, of degree p in x and p in y, symmetric (i.e., $g_p(x,y)=g_p(y,x)$), regular everywhere on $V\times V$ except on the diagonal (i.e., for $x=y$), and infinite of order $n-2$ on the latter. For an arbitrary form α of degree p, write $\beta(x)=\int_V g_p(x,y)\alpha^*(y)$; this associates with every α a form $\beta=G\alpha$ of the same degree. It is shown that the operator G is self-adjoint and commutes with $*$ and with d, hence with δ and Δ. Putting $H\alpha=\alpha-\Delta G\alpha$, $H\alpha$ is the projection (in the sense of the scalar product) of α on the (finite-dimensional) linear variety of the harmonic forms on V. Furthermore, the intersection-number (Kronecker index) of a p-chain and an $(n-p)$-chain on V is expressed by an integral formula, involving Green's form and the harmonic forms, the integrals being extended to the two chains and their (combinatorial) boundaries; this fundamental result (proved by induction on p, using the inductive definition of the Kronecker index) may be considered as a wide generalization (except for the fact that V is here taken as compact) of the Gauss integral expressing the looping coefficients of two curves, and similar results. This, combined with the Poincaré duality for V, enables the author to complete the proof of Hodge's theorem (existence of a harmonic form with assigned periods) without using the theorem of his own thesis (existence of differential forms with assigned periods). In a final section, the equation $\Delta\mu=c\mu$ for forms is studied with the help of Green's form; Fourier expansions for forms on V, in terms of eigenfunctions of Δ, are defined, and shown to be uniformly convergent for forms of class C^r, $r\geqq 2[n/4]+2$. This includes as a special case the known expansions for functions on compact Lie groups and homogeneous spaces. *A. Weil* (Chicago, Ill.).

Lieber, A. E. On the immersion of Riemannian spaces of constant curvature in one another. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 291–293 (1947).

A Riemannian space S_n of constant curvature K is spherical, flat or hyperbolic according as K is positive, zero or negative. Let S_m be a space of constant curvature in which S_n is immersed. The author obtains the following minimum values for m in the respective cases:

S_n	S_m	m
Hyperbolic	Spherical or flat	$2n-1$
Spherical	Hyperbolic or flat	$n+1$
Flat	Hyperbolic	$n+1$
Flat	Spherical	$2n-1$.

H. S. Ruse (Leeds).

Thomas, T. Y. Simplification of a differential equation due to G. Darboux. Univ. Nat. Tucumán. Revista A. **5**, 105–112 (1946).

Darboux has established the partial differential equation which must be satisfied by the functions $\phi^i(x^1,x^2)$ in order that the surface $y^i=\phi^i(x^1,x^2)$ can be isometrically mapped on a given Riemannian space R_3 [Leçons sur la Théorie Générale des Surfaces, vol. 3, Paris, 1894, p. 254]. This equation, as given by Darboux, occupies a considerable portion of an entire page. Using the procedures of tensor analysis, the author derives it in the simplified form $|\phi_{\alpha\beta}|=(1-g^{\alpha\beta}\phi_\alpha\phi_\beta)|g_{\alpha\beta}|K$, where $g_{\alpha\beta}$ are the components of the metric tensor of R_2, K the Gaussian curvature and ϕ_α, $\phi_{\alpha\beta}$ covariant derivatives. *M. Haimovici* (Jassy).

Kaplan, N. The problem of equivalence. Abh. Sem. Vektor- und Tensoranalysis [Trudy Sem. Vektor. Tenzor. Analizu] **5**, 284–289 (1941). (Russian) [MF 15616]

Eisenhart has shown how a certain set of vectors and symmetric tensors (which satisfy the differential equations of Gauss, Codazzi and Ricci) determine a V_m in a R_n [L. P. Eisenhart, Riemannian Geometry, Princeton University Press, 1926, p. 192]. The author finds necessary and sufficient conditions for the equivalence of two V_m given by two such sets of vectors and tensors. *G. Y. Rainich.*

Vranceanu, G. Sur les espaces à connexion conforme. Disquisit. Math. Phys. **1**, 63–81 (1940).

On sait qu'il existe deux points de vue principaux pour l'étude des généralisations de la géométrie conforme, celui de É. Cartan et celui de H. Weyl et de l'école de Princeton. Le premier point de vue donne naissance à la géométrie des espaces à connexion conforme, à l'aide de la notion d'espace conforme tangent en un point et par l'emploi de coordonnées hypersphériques. Le second consiste plutôt en l'étude directe des classes d'espaces riemanniens admittant des représentations conformes les uns sur les autres et utilise, comme instrument géométrique, des connexions

affines [K. Yano, J. Fac. Sci. Imp. Univ. Tokyo 4, 1–60 (1938)]. C'est à ce second point de vue que se rattache en fait le présent travail. Dans une première partie, l'auteur reprend l'étude des espaces riemanniens V_n admettant des représentations conformes les uns sur les autres par la méthode des congruences. Il retrouve le tenseur de courbure conforme W^n_{bcd} de Weyl et montre que, pour les espaces tels que $W^n_{bcd} \neq 0$, on peut construire un système complet d'invariants conformes en cherchant les invariants métriques d'un V_n pour lequel le carré du tenseur de Weyl a été réduit à l'unité par transformation conforme. La méthode ainsi dégagée est utilisée, dans une seconde partie, pour la recherche des invariants conformes "rigides" des espaces non holonomes $V_n{}^p$ (invariants communs à $V_n{}^p$ et à l'espace non holonome complémentaire V_n^{n-p}). La construction d'un système complet d'invariants conformes rigides s'effectue encore, dans le cas général, par réduction à la recherche d'invariants métriques d'un $V_n{}^p$ convenablement choisi dans la classe considérée. Il en est du moins ainsi lorsque parmi les tenseurs de torsion "conforme" de $V_n{}^p$ et ses tenseurs euleriens, il existe deux tenseurs non identiquement nuls.

A. Lichnerowicz (Strasbourg).

Yano, K., et Petrescu, St. Sur les espaces métriques non holonomes complémentaires. Disquisit. Math. Phys. 1, 191–246 (1940).

Ce papier contient un travail d'ensemble sur les espaces non holonomes plongés dans un espace X_n à connexion euclidienne avec torsion. Après quelques préliminaires classiques sur la définition des espaces non holonomes X^p_n et sur la notion d'espace non holonome X^{n-p}_n, complémentaire de X^p_n, les auteurs étudient la connexion induite par X_n dans X^p_n et X^{n-p}_n, ainsi que les tenseurs de courbure correspondants. Un chapitre relatif à la théorie des courbes appartenant à un X^p_n comprend l'étude des courbes autoparallèles (qui généralisent les géodésiques de Synge d'un espace V^p_n plongé dans un espace riemannien V_n); celle des formules de Frenet et celle des asymptotiques d'ordre $m(<p)$. Un espace X^p_n est dit totalement géodésique si chaque géodésique de X^p_n est aussi une géodésique de X_n; il est dit un hyperplan géodésique si tout vecteur tangent à X^p_n, déplacé par parallélisme dans X_n, donne encore un vecteur tangent à X^p_n. Le X^{n-p}_n complémentaire est naturellement lui aussi un hyperplan géodésique. Des conditions nécessaires et suffisantes pour qu'un X^p_n soit totalement géodésique ou, plus particulièrement, soit un hyperplan géodésique sont indiquées. Le mémoire se termine par une interprétation à l'aide d'un X^p_n totalement géodésique des théories unitaires du champ d'Einstein-Mayer et de Michal-Botsford. Un premier résultat de ce type avait été obtenu par Vranceanu [J. Physique Radium (7) 7, 514–526 (1936)].

A. Lichnerowicz (Strasbourg).

Petrescu, St. Sur un problème d'existence des espaces non holonomes $A_n{}^m$ totalement géodésiques, dans un espace A_n à connexion affine symétrique. Disquisit. Math. Phys. 4, 117–130 (1945).

K. Yano et l'auteur ont formé les équations exprimant les conditions nécessaires et suffisantes pour qu'un espace non holonome A^m_n (au sens de Vranceanu) soit totalement géodésique dans A_n [voir le mémoire analysé ci-dessus, pp. 234–235]. L'auteur utilise ces équations pour chercher à quelles conditions il existe dans A_n une famille d'espaces A^m_n totalement géodésiques dépendant de $n(n+1)(n-m)/2$ constants arbitraires. L'une des conditions obtenues exprime que le tenseur de courbure projective de Weyl doit être identiquement nul pour un tel A_n.

A. Lichnerowicz.

Kavaguči, A. [Kawaguchi]. Space of n dimensions with a connection depending on m-dimensional plane elements. Abh. Sem. Vektor- und Tensoranalysis [Trudy Sem. Vektor. Tenzor. Analizu] 5, 290–300 (1941). (Russian) [MF 15617]

The paper [dated December 1935] is devoted to a modification of an earlier theory of the author [Monatsh. Mat. Phys. 43, 289–297 (1936)] resulting from the replacement of postulate IV of that theory by a weaker one.

G. Y. Rainich (Ann Arbor, Mich.).

Norden, A. On special geometric nets in nonmetric geometry. Abh. Sem. Vektor- und Tensoranalysis [Trudy Sem. Vektor. Tenzor. Analizu] 5, 226–245 (1941). (Russian) [MF 15612]

The paper is devoted to a generalization of the theory of nets developed by J. Dubnov for metric surfaces to the case of nonmetric two-dimensional geometries. The author generalizes the concept of the Chebyshev tensor; proves the existence of a transformation of the affine connection which preserves geodesics and makes the Chebyshev tensor vanish; introduces the concept of a Cartesian net, as one which is simultaneously a Chebyshev and a geodesic net and proves that such nets can exist only in a geometry of Weyl; this leads to a discussion of these geometries. For the metric situation Dubnov had proved that in general the Chebyshev tensor determines its net but that there are exceptional cases; this is also generalized to nonmetric geometry; it is proved that such nets (called Dubnov nets) can exist only in a quasi-projective geometry, i.e., a geometry whose geodesics admit of a mapping on an isogonal field of lines in a plane. In a concluding section a surface of a three-dimensional projective space is considered from the above point of view.

G. Y. Rainich (Ann Arbor, Mich.).

Norden, A. Inner geometry of surfaces in the space of the biaxial group. C. R. (Doklady) Acad. Sci. URSS (N.S.) 55, 195–197 (1947).

The author constructs a geometry whose fundamental group is isomorphic to the subgroup of the projective transformations leaving a linear congruence of the elliptical type invariant. Since this group is completely characterized by the fact that it transforms into itself each of two conjugate axes of the congruence or the absolute lines of the space, this group is therefore a special case of the bi-axial group. The author constructs for such a space a unique affine connection with respect to which parallelism is absolute. He calls such a connection quasi-Euclidean. Such a connection is a particular Weyl connection (one arising from a Weyl metric $\nabla_k g_{ij} = \omega_k g_{ij}$). A Euclidean surface is said to be of zero curvature if it has a conjugate isotropic net and the author shows that for a quasi-Euclidean surface a coordinate system exists in which $\omega_1 = -\partial\theta/\partial y$, $\omega_2 = \partial\theta/\partial x$; if the surface is to be of zero curvature it is necessary and sufficient that θ is a harmonic function.

M. S. Knebelman.

Davies, E. T. The theory of surfaces in a geometry based on the notion of area. Proc. Cambridge Philos. Soc. 43, 307–313 (1947).

In Cartan's geometry based on the notion of area the Euclidean connection is defined in the space of the elements

each formed by a point and a bivector through the point. When the space is three-dimensional, a theory of curves can be developed by taking the bivector to be the normal bivector at each point of the curve. The author considers the corresponding problem in an n-dimensional space whose Euclidean connection depends on an element of support which is either a covariant r-vector density of weight $-p$ or a contravariant r-vector density of weight p. Formulas for the induced geometry on the subspaces are established.

S. Chern (Shanghai).

NUMERICAL AND GRAPHICAL METHODS

Katz, Alexander. Table of the first 256 powers of 2. Riveon Lematematika 1, 83–85 (1947). (Hebrew)

This is the largest published table of powers of 2. The exact values are given. *D. H. Lehmer* (Berkeley, Calif.).

***Tables of the Bessel Functions of the First Kind of Orders Four, Five, and Six, by the Staff of the Computation Laboratory.** The Annals of the Computation Laboratory of Harvard University, vol. V. Harvard University Press, Cambridge, Mass., 1947. xiii+650 pp. $10.00.

***Tables of the Bessel Functions of the First Kind of Orders Seven, Eight, and Nine, by the Staff of the Computation Laboratory.** The Annals of the Computation Laboratory of Harvard University, vol. VI. Harvard University Press, Cambridge, Mass., 1947. ix+650 pp. $10.00.

[For reviews of previous volumes cf. these Rev. 8, 406.] Volume V contains tables of $J_n(x)$ for $n=4, 5, 6$; volume VI for $n=7, 8, 9$. In both volumes x runs from 0 to 25 at intervals of 0.001 and from 25 to 99.99 at intervals of 0.01, and the functional values are given to 10 decimals. The tabulated values were computed on the automatic sequence controlled calculator from the values of $J_0(x)$ and $J_1(x)$ by means of the recurrence relations. The introduction to volume V contains a note, by R. M. Bloch, on the interpolation of the tables by means of Taylor series. *A. Erdélyi.*

Pearcey, T. The structure of an electromagnetic field in the neighbourhood of a cusp of a caustic. Philos. Mag. (7) 37, 311–317 (1946).

The author communicates results of a mathematical and numerical study of the field structure near a line focus of a cylindrical wave train possessing any finite amount of first-order aberration. The increase of the field near a cusp, above the equivalent free-space value of a plane wave, is expressed in terms of the "cusp function"

$$I(X, Y)=\int_{-\infty}^{\infty}\exp\{i(Yt+Xt^2+t^4)\}dt,$$

where the parameters X and Y are proportional to the Cartesian coordinates of the field point with reference to the center and the axis of the cusp, and depending on the wave length applied and on the shape of the caustic near the cusp. Contours of the modulus and of the phase of $I(X, Y)$ are given in two diagrams for $-8\leqq X\leqq 8$, $0\leqq Y\leqq 8$. [The author gives an incorrect reference to earlier work of Picht; he apparently intended to cite Ann. Physik (4) 80, 491–508 (1926).] *C. J. Bouwkamp* (Eindhoven).

Cavallaro, Vincenzo G. Notes sur l'approchée représentation et construction de la série Φ_6, de quelques séries hyperboliques et exponentielles, de séries ayant les nombres euleriens et bernoulliens, de nombres transcendants remarquables. Bull. Sci. École Polytech. Timişoara 11, 35–41 (1943).

$\Phi_6=\zeta(6)$, where $\zeta(s)$ is the Riemann zeta function.

Upadhyay, S. D. On interpolation formulae in two variables with reciprocal differences. Bull. Calcutta Math. Soc. 38, 156–160 (1946).

Defining operations ρ and σ by

$$\rho f(x, y)=w/[f(x+w, y)-f(x, y)]$$

and

$$\sigma f(x, y)=w/[f(x, y+w)-f(x, y)],$$

the author considers the result of successive applications of these operations of the form $\alpha_1\alpha_2$, $\alpha_1\alpha_2\alpha_3$ and $\alpha_1\alpha_2\alpha_3\alpha_4$, where each α_i is either ρ or σ, and determines conditions on the function $f(x, y)$ so that the result of such a sequence of operations shall not be altered by interchanging ρ and σ in the sequence. *T. N. E. Greville* (Washington, D. C.).

Garfath, H. L. A note on a formula of Newton and an extension thereto. J. Inst. Actuaries Students' Soc. 6, 63–66 (1946).

Instead of breaking the Taylor development for $f(a+h)$ after the linear term in h and thus getting the usual Newton formula for the approximation of a root, the author breaks it after the term in h^2. This is a well-known third degree approximation and was used by Newton himself.

E. Bodewig (The Hague).

Hruška, Václav. Lösung von Gleichungssystemen durch das Iterationsverfahren. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 44, 239–304 (1943).

The first part of the paper discusses the convergence of the iteration process. The equations are supposed to have a "canonical" form. Theorem I: Suppose that in the interval $|x-a|\leqq k_1$, $|y-b|\leqq k_2$, $|z-c|\leqq k_3$ (1) the system $F(x)=f(x, y, z)$, $G(y)=g(x, y, z)$, $H(z)=h(x, y, z)$ has a single root α, β, γ; (2) f, g, h have total differentials, while F, G, H are continuous and strictly monotonic; (3) the quotients $f_x/F'(x)$, $f_y/G'(y)$, $f_z/H'(z)$ are less than or equal to M; g_x/F', g_y/G', g_z/H' are less than or equal to N; h_x/F', h_y/G', h_z/H' are less than or equal to K, with $M+N+K<1$; (4) all roots x_i, y_i, z_i $(i=1, 2, \ldots)$ of $F(x_{i+1})=f(x_i, y_i, z_i)$, $G(y_{i+1})=g(x_i, y_i, z_i)$, $H(z_{i+1})=h(x_i, y_i, z_i)$ will lie in the interval. Then $\lim x_i=\alpha$, $\lim y_i=\beta$, $\lim z_i=\gamma$. In the case of one equation, that is, $F(x)=f(x)$, we have theorems II, III: Condition (4) is satisfied (a) if x_1 is a certain one of the endpoints of the interval $|x-a|<k$, (b) if $f'(x)F'(x)>0$ and x_1 is arbitrary within this interval. In the case of two equations we have theorem V: Condition (4) is satisfied if, in the interval, $\min F(x)<\min f(x, y)$, $\max F(x)>\max f(x, y)$, $\min G(y)<\min g(x, y)$, $\max G(y)>\max g(x, y)$ and x_1 is arbitrary in the interval. Then theorems are established on the rapidity of convergence of the process.

The second part of the paper treats the transformation of an arbitrary system to the canonical form above. In the case of one equation $\varphi(x)=0$ which has a single root α in (a, b) and for which $\varphi'(x)\neq 0$ and $\varphi(x)$ is continuous, a first form is $F(x)=F(x)+k\varphi(x)=f(x)$ where k must be chosen so that $|f'/F'|\leqq M<1$. The best choice is $k=-2/(m+\mu)$, where the positive numbers m and μ are the maximum and

minimum of φ'/F'. Then $M=(m-\mu)/(m+\mu)$. Here $F(x)$, though arbitrary, must be chosen so that $F(x)=f(x_1)$ can easily be solved and the convergence is good. The first condition is satisfied by $F(x)=x$, $1/x$, x^n, e^{cx}, $\sin x$, etc. The computation of m, μ may be avoided by taking $k=-F'(c)/\varphi'(c)$, where c lies in $[a, b]$ and as near the root as is possible. A second transformation of $\varphi(x)=0$ is $F(x)=F(x)+\rho(x)\varphi(x)=f(x)$, where $\rho(x)$ must be chosen so that $|f'/F'|\leqq M<1$ in (a, b). This is satisfied by $\rho(x)=-F'(x)/\varphi'(x)$. The process will converge better than any geometric progression and is a generalization of Newton's process, to which it reduces for $F(x)=x$.

These methods are generalized for systems of equations. The system $\varphi(x, y)=0$, $\psi(x, y)=0$ which has a single root in a rectangle is replaced by $F(x)=F(x)+\Phi(x, y)=f(x, y)$, $G(y)=G(y)+\Psi(x, y)=g(x, y)$, where $\Phi(x, y)=c_{11}\varphi+c_{12}\psi=0$, $\Psi(x, y)=c_{21}\varphi+c_{22}\psi=0$. The new system is canonical if in the rectangle the first partial derivatives of φ, ψ are continuous and not 0, the Jacobian of φ, ψ is not zero and the c_{ik} are suitably chosen. The other methods are also generalized for systems of equations. *E. Bodewig.*

Hruška, Václav. Ein weiterer Beitrag zur Lösung von Gleichungssystemen durch das Iterationsverfahren. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **44**, 399–422 (1943).

The hypotheses of theorem I of the paper reviewed above are slightly enlarged. In particular, the functions F, G, H are supposed to be merely monotonic, not strictly monotonic. In the second part the theory is applied to linear equations. A method for transforming a system of linear equations to the canonical form is indicated which, in the opinion of the reviewer, is extremely laborious in the case of more than 4 unknowns. The author ends by asserting that of all methods for solving linear algebraic equations the Gaussian method of successive elimination is the most convenient one, an opinion with which the reviewer fully agrees. *E. Bodewig* (The Hague).

Booth, A. D. Two calculating machines for X-ray crystal structure analysis. J. Appl. Phys. **18**, 664–666 (1947).

Two devices are described for carrying out certain types of crystallographic computation. The first is a multiple linkage and slider mechanism for the evaluation of

$$F(hkl)=\sum_r f_r\left\{\cos 2\pi\frac{hx_r}{a}\cdot\cos 2\pi\frac{ky_r}{b}-\cos 2\pi\frac{kx_r}{a}\cdot\cos 2\pi\frac{hy_r}{b}\right\}\cos 2\pi\frac{lZ_r}{c}.$$

The second is a simple link for evaluating

$$\sum_r f_r\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}2\pi X_r.$$

These are mechanisms designed to do particular tasks only and are justified because they perform adequately in carrying out a large volume of computation. *S. H. Caldwell.*

ASTRONOMY

van de Kamp, Peter. Two graphical procedures for evaluating the eccentricity of an astrometric double star orbit. Astr. J. **52**, 185–189 (1947).

Esclangon, Ernest. Sur l'impossibilité de transformations en satellites de la Terre, de projectiles issus de points terrestres. C. R. Acad. Sci. Paris **225**, 161–163 (1947).

Let p denote the perigeum distance of the orbit of a mass particle ejected from the surface of the Earth and encountering a resistance proportional to the velocity. The author proves that, under these circumstances, dp/dt cannot be positive; hence, unless the particle is endowed with a sufficient velocity to escape the gravitational field of the Earth, it must eventually fall back on its surface. He asserts, without proof, that these conclusions should still hold if allowance is made for the oblateness of the Earth.

Esclangon's investigation is based on oversimplified assumptions. Thus, in reality, the air resistance to a projectile is proportional to the square, rather than to the first power, of the velocity, and also to the density of the air which, in turn, depends on the altitude. Under these more appropriate conditions the motion of the projectile can be followed only by numerical integration and the validity of Esclangon's conclusions is not evident. *Z. Kopal.*

Krall, Giulio. Una nuova teoria cosmogonica fondata sul problema degli $N+1$ corpi di dimensioni finite e indipendente da ogni specificazione sulla natura dei fenomeni dissipativi. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 660–665 (1940).

Krall, Giulio. Un'importante precisazione analitica occorrente per una nuova teoria cosmogonica fondata sul problema degli $N+1$ corpi. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 666–668 (1940).

The author outlines briefly a theory, previously considered by him [Atti Accad. Naz. Lincei. Rend. (6) **15**, 664–669 (1932)] of the motions of a system subject to arbitrary internal dissipative forces. It is indicated that the system will approach asymptotically a stationary solution of minimum energy. It is shown, by an argument of "adiabatic invariance," that the asymptotic solutions thus obtained are stationary both with respect to the given differential equations and with respect to (slow) variation of auxiliary parameters on which the differential equations may depend. *W. Kaplan* (Ann Arbor, Mich.).

Agostinelli, Cataldo. Sul moto asintotico degli $n+1$ corpi. Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (2) **70**, 79–126 (1942).

This memoir reviews and extends work of Krall [cf. the preceding review] and Armellini [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 697–704 (1940); these Rev. **8**, 410] on the motion of a system of rigid bodies subject to gravitational attraction and to internal dissipative forces. It is shown that the energy must decrease with increasing time and hence that the system must approach a stationary state of minimum energy (which may be minus infinity, corresponding to a multiple collision); the total angular momentum will, however, remain constant. The problem is then to determine stationary states other than collisions. A number of such solutions are determined under particular assumptions. In general, the solutions are such that the distances between the centers of mass remain fixed; for example, a stationary solution for the case of three bodies is found such that the centers of mass are at the vertices of a triangle (reducing to the equilateral triangle of Lagrange only under special assumptions). *W. Kaplan.*

Burgers, J. M. Aerodynamical problems connected with the motion of a cloud of gas emitted by Nova Persei. I. Nederl. Akad. Wetensch., Proc. **50**, 262–271 (1947).

The paper is one of a series dealing with the hydrodynamics of motions in the interstellar gas. The special

case is considered of the expanding shell of gas surrounding Nova Persei 1901. To simplify the problem, the author analyzes the one-dimensional case of a thin flat cloud of gas, with a density equal to 1500 times that of the interstellar gas, striking suddenly upon the interstellar gas and with an initial velocity of the order of 1800 km/sec, far in excess of the velocity of sound. The gas is supposed to be monatomic hydrogen, with a ratio of specific heat $\kappa=5/3$.

The equations are written down to represent the motion of the front of the advancing cloud and of the compression wave (the shock wave at the head of the region affected by the cloud). A solution is given for the instants directly following the moment of the collision between the advancing sheet and the general interstellar medium. The derived temperature for the front of the cloud is about 100,000,000°.

B. J. Bok (Cambridge, Mass.).

Burgers, J. M. Aerodynamical problems connected with the motion of a cloud of gas emitted by Nova Persei. II. Nederl. Akad. Wetensch., Proc. 50, 332–338 (1947).

[Cf. the preceding review.] The first part of the second paper is an attempt at the construction of an asymptotic solution, valid for large values of the time. The derived temperature is now of the order of 79,000,000°.

The second part of this paper deals with the slightly different problem in which the advancing cloud of gas (similar to a piston on which a force is acting) is supposed to move with constant velocity. The derived values for the density and temperature at the front of the cloud are very similar to those found previously. *B. J. Bok.*

Burgers, J. M. Aerodynamical problems connected with the motion of a cloud of gas emitted by Nova Persei. III. Nederl. Akad. Wetensch., Proc. 50, 442–451 (1947).

Continuation of the two papers reviewed above. The author no longer considers his discussion to be completely adequate.

Milne, E. A. On the spiral character of the external galaxies. Monthly Not. Roy. Astr. Soc. 106, 180–199 (1946).

The orbits of free particles in the neighborhood of a massive nucleus of matter are calculated on the author's kinematic theory of gravitation [cf. Milne, Proc. Roy. Soc. London. Ser. A. 160, 1–23 (1937), pp. 18 ff.]. They are shown to be Keplerian orbits on the dynamical τ-time scale, but open equiangular spirals on the kinematic t-time scale. These results are invoked to explain certain features of the extragalactic spiral nebulae. Milne points out that the observed nebular arms are not envelopes of the actual orbits, but rather the loci of the present positions of the stars composing them. An equation of these spirals is derived which, the author claims, determines the order of magnitude of the size of a nebula, the past epoch at which the emission of particles from its nucleus began and the number of convolutions which should have been formed since then. The theoretical locus is found to unwind for a finite number of convolutions in the sense of the orbital motion and then to double back, unwinding thereafter for the same number of convolutions in the retrograde sense. Therefore, in the outer parts of the nebula, the sense of unwinding of its spiral arms may be retrograde as compared with the orbital motion, while in the parts closer to the nucleus this sense may be direct. In conclusion, Milne shows that certain observational data bearing on our own galactic system and on the great Andromeda nebula are not inconsistent with the consequences of his kinematic theory of gravitation.

Z. Kopal (Cambridge, Mass.).

Lyttleton, R. A. On the general equations of stellar structure. Monthly Not. Roy. Astr. Soc. 106, 228–239 (1946).

The hydrostatic equilibrium of a star in convective equilibrium is considered for the general case when radiation pressure is included without approximation. The starting point of the investigation is the relation

$$(*)\qquad \frac{T^3}{\rho}+\frac{3k}{4a\mu H(\gamma-1)}\log\left\{\frac{T\rho_0^{(\gamma-1)}}{T_0\rho^{(\gamma-1)}}\right\}=\frac{T_0^3}{\rho_0}$$

between the temperature T and the density ρ, where k is the Boltzmann constant, a is the radiation constant, γ is the ratio of the specific heats of the gas, H is the mass of the hydrogen atom, μ is the mean molecular weight and ρ_0 and T_0 are particular values of ρ and T on the adiabatic considered. In terms of the relation (*), the equations of hydrostatic equilibrium are reduced to certain nondimensional forms and the solutions (valid near the center) are obtained as series expansions. The coefficients of the various series are tabulated for different values of the ratio of the gas pressure to the radiation pressure at the center of the configuration. [A relation essentially equivalent to (*) was first obtained by J. Wasiutynski, Monthly Not. Roy. Astr. Soc. 100, 362–369 (1940). Also the investigation of the equations of equilibrium overlaps to a large extent the earlier work of L. R. Henrich, Astrophys. J. 93, 483–501 (1941).] *S. Chandrasekhar* (Williams Bay, Wis.).

Ledoux, P. Stellar models with convection and with discontinuity of the mean molecular weight. Astrophys. J. 105, 305–321 (1947).

A discontinuous outward decrease in the mean molecular weight μ is shown to produce a convective instability. The resulting convection rapidly smooths out the discontinuity, leaving a transition zone in which $\mu\propto M(r)P^{7/8}$. Models including such a transition zone differ relatively little from those obtained by assuming a discontinuous change in μ, and no convection. *T. G. Cowling* (Bangor).

Proisy, Paul. Variation de la loi d'assombrissement d'une radiation monochromatique à l'intérieur de l'atmosphère solaire. Cas où l'équilibre radiatif est réalisé. C. R. Acad. Sci. Paris 224, 1334–1336 (1947).

The law of surface darkening is assumed known as a polynomial in $\cos\theta$, where θ is the angle between the radius and the line of sight. The emissivity and opacity are also assumed known. The angular distribution of radiation is calculated at a general depth below the surface, taking into account the sun's curvature. *T. G. Cowling* (Bangor).

Sorokina, A. On stability of gaseous spheres. C. R. (Doklady) Acad. Sci. URSS (N.S.) 54, 677–680 (1946).

The energy-increase due to a disturbance from equilibrium is taken to be $\frac{1}{2}\int\delta\rho\delta\phi dV+\frac{1}{2}\int(c_0^2/\rho_0)(\delta\rho)^2dV$, where c_0 is the local speed of sound, ρ_0 the undisturbed density, and $\delta\rho$, $\delta\phi$ the disturbances in density and gravitational potential. The first integral is normalized to unity and that disturbance is found which makes the second integral a minimum. The sphere is found to be stable for all nonradial disturbances; for radial disturbances, an Emden sphere is stable if its index is less than 3. [In the first integral, $\delta\rho$ is apparently taken at a particular point; in the second, following a particular element of mass. In any case, the expression for the energy-increase is incorrect with either meaning of $\delta\rho$.] *T. G. Cowling* (Bangor).

RELATIVITY

* **Johnson, Martin. Time, Knowledge, and the Nebulae. An Introduction to the Meanings of Time in Physics, Astronomy, and Philosophy, and the Relativities of Einstein and Milne.** Dover Publications, New York, 1947. 189 pp. $4.95.

The author discusses the meaning of time in physics and in philosophy. The concept of time is discussed as it arises in special and general relativity and in cosmology. A good deal of space is devoted to Milne's kinematical relativity and to his two time-scales. The foreword is written by E. A. Milne. *A. Schild* (Princeton, N. J.).

Wilson, W. Kinematic relativity. Philos. Mag. (7) **37**, 421–426 (1946).

The author derives the equation of motion of a free particle in Milne's kinematic relativity by use of Minkowski space-time with imaginary time coordinate. *A. Schild.*

Band, William. A critical examination of Milne's kinematical relativity. Philos. Mag. (7) **37**, 551–563 (1946).

The author shows that Milne's formulae for the velocity distribution in a substratum and for the acceleration of a test particle are special cases of more general equations which satisfy the requirements of Lorentz invariance (equivalence) and of spatial isotropy. Published comments on Milne's theory and Milne's criticism of general relativity are reviewed. *A. Schild* (Princeton, N. J.).

Garín de Alvarez, Manuela. Decomposition of the matrix of the general Lorentz transformation into simple factors. Bol. Soc. Mat. Mexicana **3**, 27–35 (1946). (Spanish)

Decomposition of the general Lorentz transformation L into the form $L=ABC$, where the matrices A, C represent rotations of axes in the ordinary sense of analytical geometry, and B a special Lorentz transformation (one affecting the time-variable and only one of the space-variables). *H. S. Ruse* (Leeds).

Hessaby, M. Continuous particles. Proc. Nat. Acad. Sci. U. S. A. **33**, 189–194 (1947).

A reinterpretation of Maxwell's equations in general relativity, in which particles appear as a continuous energy distribution in space with density everywhere finite, charged particles having also finite charge density everywhere. *A. Pais* (Princeton, N. J.).

Romero Juárez, Antonio. The lunar motion in Birkhoff's theory of gravitation. Bol. Soc. Mat. Mexicana **3**, 13–19 (1946). (Spanish)

The author obtains a formula, based on Birkhoff's gravitational theory, for the advance of the perigee of a particle of infinitesimal mass (the moon) moving in the field of two other masses m, m_1 (sun and earth) and near to m_1. He finds that the advance of the perigee in one revolution of m_1 about m is approximately

$$\epsilon=2\pi\frac{m^2+3mm_1+m_1^2}{a(m+m_1)},$$

where a is the radius of the orbit of m_1, assumed circular. The corresponding formula in Einstein's relativity is $\epsilon=3\pi m/a$. The new value of ϵ is approximately two-thirds of this. *H. S. Ruse* (Leeds).

Gião, Antonio. Sur la propagation de la lumière dans un champ électrostatique. C. R. Acad. Sci. Paris **224**, 1212–1214 (1947).

In previous work by the author [Portugaliae Phys. **2**, 1–98 (1946); Portugaliae Math. **5**, 145–193 (1946); these Rev. **8**, 121, 555] electromagnetic phenomena were assumed determined by the external metric (1) $d\Omega^2=\omega_{ik}dx^idx^k$ of space-time just as gravitational phenomena are supposed determined by the internal metric (2) $ds^2=g_{ik}dx^idx^k$. Rays of light are null geodesics for (1), and for (2) if space-time has constant mean curvature. A formula is derived for the deviation of light rays in a spherically symmetrical electrostatic field analogous to that produced by a gravitational field. *C. Strachan* (Aberdeen).

Snyder, Hartland S. The electromagnetic field in quantized space-time. Physical Rev. (2) **72**, 68–71 (1947).

In a previous paper [Physical Rev. (2) **71**, 38–41 (1947); these Rev. **8**, 412] the author gave an example of a Lorentz-invariant space-time whose space coordinates are noncommuting Hermitian operators with discrete spectra, thus consistent with a natural unit of length. Here, for this space-time, relativistically invariant equations for the electromagnetic field are given along with charge-conservation and energy equations, scalar and vector potentials, etc., the field quantities being operators. Partial derivatives of a field quantity with respect to a coordinate are replaced by i times the commutator of the corresponding component of the momentum-energy vector with the operator corresponding to the field quantity. Operators satisfying the resulting field equations are found and a process is established analogous to the Fourier analysis of field quantities in continuous space-time. *C. Strachan* (Aberdeen).

Narlikar, V. V., and Vaidya, P. C. A spherically symmetrical non-static electromagnetic field. Nature **159**, 642 (1947).

The authors claim to have obtained the most general non-static spherically symmetric solution of the relativistic equations of a free electromagnetic field. *A. Schild.*

MECHANICS

Pailloux, Henri. Sur une extension à l'espace de la formule de Savary. C. R. Acad. Sci. Paris **224**, 1539–1540 (1947).

A surface of a solid rolls upon an applicable surface of another solid so that only corresponding points of the surfaces are brought into contact. An arbitrary surface carried by the rolling solid moves, therefore, with two degrees of freedom. The envelope of this arbitrary surface is considered and its fundamental formulas derived. *M. Goldberg* (Washington, D. C.).

Cârstoiu, I. Sur le mouvement d'une plaque rigide dans son propre plan. Bull. Sci. École Polytech. Timişoara **11**, 263–265 (1944).

Lee, Hwa-Chung. The universal integral invariants of Hamiltonian systems and application to the theory of canonical transformations. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 237–246 (1947).

In this note the author considers the integral invariants which have all Hamiltonian systems in common. Such a

system can be written as

$$\frac{dx^\alpha}{dt}=\epsilon^{\alpha\beta}\partial_\beta H;\quad \alpha=1,\cdots,2n;\quad \epsilon^{\alpha\beta}=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix};$$

$$x^i=q^i\ (i=1,\cdots,n),\quad x^{n+i}=p_i.$$

It is proved that $\int A_{\alpha_1\cdots\alpha_r}\delta x^{\alpha_1}\cdots\delta x^{\alpha_r}$, where $A_{\alpha_1\cdots\alpha_r}$ is skew symmetric, is an absolute integral invariant if and only if $r=2m$ is even and

$$A_{\alpha_1\cdots\alpha_{2m}}=c\epsilon_{[\alpha_1\alpha_2}\cdots\epsilon_{\alpha_{2m-1}\alpha_{2m}]}.$$

From this it follows that there is only one universal relative integral invariant of odd order $2s-1$ and that there are no universal relative integral invariants of even order. Conversely it is proved that a system which has one of these integral invariants is necessarily a Hamiltonian system. An immediate consequence of this theorem is that a transformation which carries every Hamiltonian system into a Hamiltonian system must satisfy the necessary and sufficient condition $\sum\bar{p}_i\delta q^i=c\sum p_i\delta q^i+\delta w$. The new Hamiltonian function is $\bar{H}=cH-\partial w/\partial t+\bar{p}_i\partial q^i/\partial t$. For $c=1$ this transformation is a contact transformation. *J. Haantjes.*

Kasner, Edward. The inverse problem of dynamics. Univ. Nac. Tucumán. Revista A. **5**, 95–104 (1946).

The problem under discussion is the following. A particle moves in a plane in a positional field of force; it is required to determine the field of force from a knowledge of a subset of the totality of ∞^3 possible trajectories. The principal result is that in general the field of force in a region is determined, to within an arbitrary constant factor, when four one-parameter subfamilies of trajectories are given, such that through each point of the region there pass four curves, one from each subfamily, with distinct tangents. Except for a few new remarks concerning exceptional cases, the results have all been given in the author's colloquium lectures [Amer. Math. Soc. Colloquium Publ., v. 3_{II}, New York, 1913]. *L. A. MacColl* (New York, N. Y.).

Hydrodynamics, Aerodynamics

García, Godofredo. General laws of the variation of the density in the nebulae formed in general by viscous fluids and in particular by a perfect fluid in motion. Actas Acad. Ci. Lima **9**, 221–265 (1946). (Spanish)

García, Godofredo. General and complete cardinal and scalar equations of relative rotational motion of viscous compressible fluids. Actas Acad. Ci. Lima **9**, 271–304 (1946). (Spanish)

García, Godofredo. The fundamental equations of hydrodynamics in generalized Lagrangian form. Actas Acad. Ci. Lima **9**, 304–329 (1946). (Spanish)

The author considers the problem of motion of a general fluid subject to internal gravitational forces. Various special assumptions on the character of the motion (e.g., irrotationality, slowness) and on the relation between pressure and density are made, the corresponding differential equations are obtained and methods of solution are indicated. This is a continuation of earlier work of the author [cf. Revista Ci., Lima **46**, 3–11 (1944); these Rev. **6**, 75] and of Levi-Civita [Scritti Mat. Off. a L. Berzolari, Pavia, 1936, pp. 161–168]. *W. Kaplan* (Ann Arbor, Mich.).

Kaplan, Carl. The flow of a compressible fluid past a curved surface. Wartime Rep. Nat. Adv. Comm. Aeronaut., no. **L-320**, 74 pp. (4 plates) (1943).

An iteration method proposed by Ackeret is used to calculate the flow over a thin two-dimensional bump in a steady compressible stream. In this procedure the velocity potential is expanded in a series of ascending powers of the thickness ratio: $\phi(x,y)=x+t\phi_1(x,y)+t^2\phi_2(x,y)+\cdots$. By substitution in the nonlinear potential equations it is found that the equation for $\phi_1(x,y)$ is the customary Prandtl-Glauert one and that each successive $\phi_i(x,y)$ is given by a linear partial differential equation of the form

$$(1-M^2)\phi_{i_{xx}}+\phi_{i_{yy}}=F(\phi_{i-1},\phi_{i-2},\cdots,\phi_1),$$

where M is the Mach number of the undisturbed stream. Thus the problem is solved by iteration. For each $\phi_i(x,y)$, the boundary condition of tangential flow at the solid surface is satisfied to the same power of t as is involved in the expression for $\phi(x,y)$.

The solid boundary involved here is a symmetrical shape with cusps front and rear. The maximum height of the bump is 1/20 of the distance between the cusps. The iteration is carried through the determination of $\phi_3(x,y)$. The numerical results are presented for $M=0$, .50, .75 and .83. At $M=.83$, there is a considerable region of shockless supersonic flow near the bump. Pressure coefficients are compared with those obtained according to the Prandtl-Glauert and von Kármán-Tsien theories. Certain comparisons are also made with the Poggi method of expansion in powers of M^2, which is applied to the curved bump in an appendix.

The convergence of the iteration process is not proved. The author believes that divergence will occur when the "upper critical" Mach number is reached; i.e., the Mach number at which the "limit line" appears in calculations performed by the hodograph transformation [see, for example, von Kármán, J. Aeronaut. Sci. **8**, 337–356 (1941); these Rev. **3**, 220]. *W. R. Sears* (Ithaca, N. Y.).

Perl, W. Calculation of compressible flows past aerodynamic shapes by use of the streamline curvature. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1328**, 38 pp. (49 plates) (1947).

Bilharz, Herbert, and Hölder, Ernst. Calculation of the pressure distribution on bodies of revolution in the subsonic flow of gas. I. Axially symmetrical flow. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1153**, 27 pp. (4 plates) (1947).

Translation of Zentrale für Wissenschaftliches Berichtswesen über Luftfahrtforschung, Forschungsber. no. **1169/1** (1940).

Nekrasov, A. I. Flow past a Joukovsky aerofoil with a source and a sink on its contour. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 41–54 (1947). (Russian. English summary)

That a source and a sink on the contour of an aerofoil may increase its lift force is now well known. This phenomenon is treated mathematically in the present paper for a profile obtained by the Joukovsky method.

From the author's summary.

Ringleb, F. Some aerodynamic relations for an airfoil in oblique flow. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1158**, 15 pp. (5 plates) (1947).

[Translation of Zentrale für Wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters,

Forschungsber. no. 1497 (1941).] Some aerodynamic relations are derived which exist between two infinitely long airfoils if one is in a straight flow and the other in oblique flow, and both present the same profile in the direction of flow. *Author's summary.*

Miles, John W. The aerodynamic forces on an oscillating airfoil at supersonic speeds. J. Aeronaut. Sci. 14, 351–358 (1947).

The author transforms the linearized equation for the velocity potential in unsteady supersonic flow to the form of the wave equation in an $(\bar{x}, y, z, \bar{t})$-space. A fundamental solution is that for a point source $\phi=(1/r)f(\bar{t}\pm r/c)$, $r^2=\bar{x}^2+y^2+z^2$. Selecting $f(x)$ as $\exp(i\omega\mu x)$, where ω is constant and μ is a parameter specifying the stream Mach number, and returning to the physical (x, y, z, t)-space, he obtains the potential due to a periodically fluctuating source at $(\xi, \eta, 0, t)$. As in the corresponding steady case, the source produces a non-vanishing velocity component $w(x, y, 0, t)$ in the (x, y)-plane only at the point $x=\xi$, $y=\eta$, where it is proportional to the source strength. Thus the problem of the oscillating airfoil can be treated by distributing such sources and the pressure at any point of the airfoil is found by integration over the Mach cone forward of the point.

The cases treated in detail here are those of the infinite plane airfoil in plunging (vertical translation) and pitching (rotation about the mid-chord) oscillations. Lift and pitching moment are calculated in both cases. The author states that his formulae agree with those of v. Borbély [Z. Angew. Math. Mech. 22, 190–205 (1942); these Rev. 5, 136] and his numerical results with those of Possio [Aerotecnica 18, 441–458 (1938)]. The present numerical results are plotted for stream Mach numbers of 1.2, $\sqrt{2}$, 1.6 and 2, and for reduced frequencies k from 0 to 1.5. In appendices, formulae are developed for use where k is very small, and also asymptotic expressions for large Mach numbers. *W. R. Sears.*

Struminsky, V. V. Glissement d'une aile dans un gaz visqueux compressible. C. R. (Doklady) Acad. Sci. URSS (N.S.) 54, 765–768 (1946).

The Navier-Stokes equations and the equation of energy are written for the flow of a compressible viscous fluid about a yawed infinite cylinder, i.e., flow in which all physical quantities are independent of the coordinate z measured along the cylindrical axis. In certain cases (isothermal, adiabatic) the flow pattern in planes normal to the axis is independent of the axial flow; hence flow patterns can be carried over from plane cases. In such cases the forces on the cylinder will depend upon the Mach and/or Reynolds numbers based on the normal (to the axis) component of the stream velocity and on the angle of yaw in a simple way.

For large Reynolds numbers, the general equations are approximated in the manner of Prandtl's boundary-layer theory and it is further assumed that the Prandtl number is unity and there is no heat transfer from the cylinder. The subsequent treatment follows closely that of Dorodnitzyn [same C. R. (N.S.) 34, 213–219 (1942); these Rev. 4, 176]. Again the energy equation can be integrated and becomes a statement of constant enthalpy. The same independent variables are introduced and in this case a pair of simultaneous nonlinear partial differential equations results. The author proposes to obtain approximate solutions by the same method as Dorodnitzyn, using the Kármán-Pohlhausen method, assuming quartic expressions for both the axial and normal velocity components.

In the special case of the yawed flat plate, the problem can be related to the corresponding plane-flow problem, which was treated in detail by Dorodnitzyn [loc. cit.]. After correction of an apparent typographical error in equations (13), the results appear to state that the boundary-layer flow is in the free-stream direction and proportional to $\varphi_0'(\tau)$ [see these Rev. 4, 176], where, in this case, τ is defined proportional to the normal component of the stream velocity. For other cylinders, the boundary-layer flow will deviate from the stream direction, assuming an axial direction at the point of separation of the normal flow.

W. R. Sears (Ithaca, N. Y.).

Evvard, John C. Distribution of wave drag and lift in the vicinity of wing tips at supersonic speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1382, 28 pp. (5 plates) (1947).

Moeckel, W. E., and Connors, J. F. Charts for the determination of supersonic air flow against inclined planes and axially symmetric cones. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1373, 17 pp. (18 plates) (1947).

Lighthill, M. J. A mathematical method of cascade design. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2104 (8766), 18 pp. (1945).

It is by now well known that separation of a body's boundary layer is associated with large positive pressure gradients along the surface. Hence a method of design, which will prevent large positive pressure (negative velocity) gradients, will probably also prevent separation, so that the flow will adhere as closely as possible to theoretical predictions and the aerodynamic losses be minimised. The method of this paper, proceeding by means of the conformal transformation of a cascade of aerofoils into a circle, enables the shape of the aerofoils to be determined from any preassigned distribution of velocity (which satisfies certain conditions) given on the circle. Study of examples combined with a knowledge of practical requirements will assist designers to choose this distribution satisfactorily. One example is worked through in great detail in the paper to show how the method is carried out in practice.

Author's summary.

Tetervin, Neal. A review of boundary-layer literature. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1384, 36 pp. (4 plates) (1947).

Hayes, Wallace D. On hypersonic similitude. Quart. Appl. Math. 5, 105–106 (1947).

Tsien in a recent paper [J. Math. Phys. Mass. Inst. Tech. 25, 247–251 (1946); these Rev. 8, 237] has discussed hypersonic flows around slender bodies and has pointed out that the product of Mach number and fineness ratio is a basic similarity parameter. The author enlarges on this notion, indicating that the problem of hypersonic flow about a slender body in three dimensions is the same as that of a certain two-dimensional nonsteady flow (with time replacing the lengthwise spatial coordinate) characterized by essentially the same similarity parameter. *D. P. Ling.*

Tsien, Hsue-shen. Flow conditions near the intersection of a shock wave with solid boundary. J. Math. Phys. Mass. Inst. Tech. 26, 69–75 (1947).

The author considers a two-dimensional flow along a boundary at a point where the slope of the boundary is continuous but where the radius of curvature may have a

finite discontinuity. He supposes that a shock wave meets the boundary at this point, intersecting it normally (since the flow direction is not to be abruptly altered). Using the equations of motion and of continuity, and the normal shock conditions, he demonstrates that the Mach numbers before and after the shock at the boundary surface (there is no question of a boundary layer here) are determined by the ratio of the radii of curvature of the boundary at the corresponding points. The results are given in specific formulas and graphs. He then points out that the pressure gradient along the boundary reverses its sign on passing through the shock, a phenomenon which had been observed previously in the course of experiment and calculation.

D. P. Ling (Murray Hill, N. J.).

Thomas, T. Y. On curved shock waves. J. Math. Phys. Mass. Inst. Tech. **26**, 62–68 (1947).

The author derives compact formulas for velocity components, pressure, density and the first partial derivatives of these quantities just downstream of a curved shock wave. The procedure is carried through in detail only for two-dimensional flows, but the author points out that it is readily generalized. In the two-dimensional case, the quantities mentioned above (and the curvature of the streamlines just downstream of the shock) are expressed in terms of the corresponding quantities on the upstream side and of the orientation and curvature of the shock line. Detailed formulas are given for the case where the incident flow is uniform. *D. P. Ling* (Murray Hill, N. J.).

Hicks, Bruce L., Montgomery, Donald J., and Wasserman, Robert H. The one-dimensional theory of steady compressible fluid flow in ducts with friction and heat addition. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1336**, 26 pp. (1947).

de Haller, P. The application of a graphic method to some dynamic problems in gases. Sulzer Tech. Rev. **1945**, no. 1, 6–24 (1945). [MF 14076]

The method of characteristics is applied to a number of fluid flow problems with partial differential equations of hyperbolic type. In the case of one-dimensional flow, a variety of boundary conditions are handled and the corresponding graphical constructions are given in some detail. These basic constructions are then applied to the solution of several special problems involving the interaction and reflection of rarefaction and compression waves. In the case of three-dimensional axially symmetrical supersonic flow, an adaptation of the Prandtl-Busemann method for two dimensions is used in which a correction is applied to the velocity vector in the hodograph plane. The design of a perfect nozzle and the construction of a supersonic flow past a cone are treated by this device. *E. N. Nilson.*

Sauer, R. General characteristics of the flow through nozzles at near critical speeds. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1147**, 18 pp. (2 plates) (1947).

Translation of Zentrale für Wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters, Forschungsber. no. **1992** (1944).

Gubanov, A. Elastic capillary gravitational surface waves. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **17**, 162–169 (1947). (Russian. English summary)

An equation is deduced for the velocity of propagation of waves on the surface of an elastic-viscous body. In the limiting case of small frequencies this equation gives the velocity of capillary gravitational waves on a liquid surface. For high frequencies, as another limiting case, it gives the velocity of Rayleigh waves at the surface of a solid body. The method of successive approximations is used to study cases coming close to the above two limits.

Author's summary.

Davies, D. R. Turbulence and diffusion in the lower atmosphere with particular reference to the lateral effect. Proc. Roy. Soc. London. Ser. A. **190**, 232–244 (1947).

The author shows that, by introducing a lateral component of atmospheric turbulence over plane saturated liquid surfaces, the theoretical concentration of vapor and the rate of decay of concentration with height are brought much nearer to experimental values without appreciably affecting theoretical rates of evaporation. The computations are carried out for areas of parabolic shape. *C. C. Torrance.*

Matthewman, A. G. A discussion of the pressure-tendencies associated with gradient and horizontal geostrophic flow. A formula for the variation with height of the vertical velocity. Philos. Mag. (7) **37**, 706–716 (1946).

The following assumptions are made: (1) the air is not saturated; (2) gradient wind motion prevails throughout the horizontal environment of the place for which the pressure tendency is required; (3) the motion is dry-adiabatic. The pressure tendency is derived in terms of the fields of motion, temperature and humidity. It is shown that, except in very high latitudes, horizontal geostrophic flow with an appreciable northerly or southerly component implies a pressure tendency much larger than ordinarily observed. This pressure tendency can be moderated only by assuming that there is a slight curvature of the air trajectory up or downwind, or a vertical variation of the vertical velocity, or both. With gradient flow terms of much larger magnitude than the tendency term occur in the expression for the divergence. Consequently, one obtains the following formula for the variation of the vertical velocity ω with elevation:

$$\frac{\partial \omega}{\partial z}=\frac{1}{G\kappa+\lambda}\left\{\kappa G\frac{\partial G}{\partial \rho}+G^2\frac{\partial \kappa}{\partial \rho}+\frac{2\omega\cos\varphi}{R}G_\varphi\right\},$$

where G is the gradient, κ the curvature of the trajectory (projected in the horizontal plane), λ the Coriolis parameter, ω the angular velocity of the earth's rotation, φ the latitude, R the earth's radius, G_φ the meridional component of the gradient wind (positive towards N), ρ the direction of the gradient wind. *B. Haurwitz* (New York, N. Y.).

Lineykin, P. S. Hydrodynamical theory of monsoons upon a round island. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **11**, 103–126 (1947). (Russian. English summary)

The present paper contains the investigation of the periodic process of convection in the atmosphere upon the earth's surface. The distribution of temperature upon the earth's surface is given by an equation, modeling the corresponding temperature-regime on the surface of a round island and the surrounding infinite sea. For the solution of the equation of convection the author applies the method of development of the unknown functions, i.e., velocity, pressure and temperature of air particles, into infinite power-series by small parameter for which is taken the coefficient

of the air's thermal expansion, $\alpha=0.00366$. The approximate expressions of the solutions of the differential equations of this problem allow one to establish some laws of the monsoon-field, the most important being the following. (1) The monsoon-wind reaches the atmosphere's lower layer at the height $z=\pi\sqrt{(\nu/\omega)}$, supposing the angular velocity of the atmosphere to be $\omega=5\cdot10^{-5}$ sec^{-1}, and the coefficient of eddy viscosity $\nu=3\cdot10^{4}$ cm^2/sec. Higher on the layer an "antimonsoon" with an opposite direction of the horizontal component of velocity is situated. (2) The trajectories represent in the projection upon the meridional plane a curve-family represented on a subjoined map. The successive situations of the particle on the trajectory show that the transfer of air masses in the monsoon takes place at considerable distances in comparison with the dimensions of the continent. (3) The isobar-surfaces are little different from the horizontal planes. (4) The chief part in the process of heat transfer belongs to the vertical heat transfer by the flow. The proposed theory must still be perfected.

From the author's summary.

Elasticity, Plasticity

Boulton, N. S. Plastic stresses in a semi-infinite cohesive mass due to a surcharge covering a strip of the surface of infinite length. Philos. Mag. (7) **37**, 733–746 (1946).

The paper is concerned with the following plane problem of the statics of heavy cohesive media with internal friction: a half-plane is subjected to a uniformly distributed pressure along its entire horizontal boundary and an additional pressure which is uniformly distributed over a certain segment of this boundary. L. Prandtl [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1920, 74–85] solved this problem in the absence of gravity, giving a solution in closed form. If gravity cannot be neglected, numerical integration based on the method of characteristics may be used [see V. V. Sokolovsky's "Statics of Earthy Mediums," Moscow, 1942, chaps. IV and V; these Rev. **6**, 27]. The author suggests that for sufficiently small unit weight this cumbersome numerical integration may be replaced by a perturbation procedure in which the product of the unit weight and a characteristic length divided by the yield constant is used as perturbation parameter. *W. Prager.*

Oldroyd, J. G. Two-dimensional plastic flow of a Bingham solid. A plastic boundary-layer theory for slow motion. Proc. Cambridge Philos. Soc. **43**, 383–395 (1947).

Discussing the equations of motion of an isotropic visco-plastic material (Bingham solid) the author establishes the fact that dynamical similarity in geometrically similar systems is governed by two dimensionless parameters, one of which has the nature of a Reynolds number while the other is the product of the yield value and a characteristic length divided by the product of the coefficient of viscosity and a characteristic velocity. As a rule, the inertia, viscosity and yield terms in the equations of motion are equally important. If the terms of one of these three groups predominate, however, the motion is either perfect fluid flow, classical slow motion of a viscous fluid or "a new type of flow" occurring in most parts of the plastic region. [The author does not seem to realize that this flow is the well-known flow of a perfectly plastic material obeying the laws of von Mises' theory. The reviewer discussed the possibility of considering the von Mises material as a limiting case of a visco-plastic material and pointed out the necessity of a theory of the plastic boundary layer if this point of view is adopted [Mécanique des solides isotropes au delà du domaine élastique, Mémor. Sci. Math., no. **87**, Paris, 1937, pp. 12 and 28].] The equations of the plastic boundary layer are established and integrated in the following three cases: (a) flow under uniform pressure gradient between moving parallel plates, (b) plane flow in the neighborhood of a semi-infinite plane knife which moves with indefinitely small velocity through an infinite Bingham solid at rest, (c) plane jet of plastic material forced by pressure into plastic half-plane through a gap in the rigid boundary of this half-plane. *W. Prager* (Providence, R. I.).

Oldroyd, J. G. Rectilinear plastic flow of a Bingham solid. I. Flow between eccentric circular cylinders in relative motion. Proc. Cambridge Philos. Soc. **43**, 396–405 (1947).

The rectilinear flow of a Bingham solid in the tubular space between two cylinders with parallel (but not coinciding) axes is discussed. For a viscous fluid the velocity distribution measured in terms of the relative velocity of the two cylinders is independent of the force per unit length with which the cylinders are pulled apart. For a Bingham solid with finite yield value, however, this is no longer the case. The velocity contours are shown for three values of a dimensionless parameter which is proportional to the yield value and inversely proportional to the relative velocity of the cylinders. *W. Prager* (Providence, R. I.).

Oldroyd, J. G. On a derivation of the equations of equilibrium of a thin plate. Philos. Mag. (7) **37**, 648–651 (1946).

This paper corrects an oversight in a derivation of the equations for small transverse bending of thin elastic plates by H. Jeffreys [Philos. Mag. (7) **32**, 365–368 (1941); these Rev. **3**, 224]. The oversight concerns the expression for the effective transverse force intensity at the boundary of the plate. The present paper also gives explicit expressions in terms of orthogonal curvilinear coordinates for edge bending moment and effective edge shear force, when the boundary of the plate coincides with one of the coordinate curves. *E. Reissner* (Cambridge, Mass.).

Kuntze, I. P. Stabilité des plaques comprimées satisfaisant à la théorie de plasticité de Prager. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 387–389 (1947).

An earlier form of the reviewer's theory of plasticity [Rev. Fac. Sci. Univ. Istanbul (A) **5**, 215–226 (1941); these Rev. **3**, 32] is applied to the plastic buckling of rectangular plates under edge thrusts. The following two cases are discussed: cylindrical buckling of a simply compressed plate; buckling of a simply supported plate which is subjected to a constant edge thrust all along its edge. [The results reflect the special stress-strain relation adopted in the earlier form of the theory.] *W. Prager* (Providence, R. I.).

Ling, Chih-Bing. Torsion of a circular tube with longitudinal circular holes. Quart. Appl. Math. **5**, 168–181 (1947).

This paper presents a solution of St. Venant's torsion problem of a circular tube having a ring of uniformly dis-

tributed longitudinal circular holes of equal radii. Three harmonic functions are found which are invariant with respect to rotation about the origin through an angle $2m\pi/k$, where m is an integer and k the number of circular holes within the tube wall. The boundary conditions are sufficient to determine all the parameters and the unknown constant values of the torsion function on the interior boundaries. The maximum shearing stress occurs on the boundary across the narrowest sections, but it is not necessarily on the exterior boundary unless the eccentric holes are sufficiently remote from the exterior boundary. Numerical results are shown for tubes with $k=2$ or 6, the holes having equal radii and the nearest distance between any two adjacent boundaries being equal to the radius of one hole. Some comparisons are made for relative reductions in torsion stiffness from that of the tube itself to the reduction in corresponding area. *D. L. Holl* (Ames, Iowa).

Hadji-Argyris, J., and Dunne, P. C. The general theory of cylindrical and conical tubes under torsion and bending loads. Single and many cell tubes of arbitrary cross-section with rigid diaphragms. J. Roy. Aeronaut. Soc. **51**, 199–269 (1947).

This paper deals with the determination of stresses and displacements in cylindrical and slightly conical shells. In conformity with many earlier investigations of special cases of the theory, the assumption is made that the shape of the cross section of the shell is maintained by a closely spaced system of diaphragms rigid in their own plane and parallel to the root section. This assumption makes it possible to express the stresses in the shell in terms of four displacements, three of which are rigid body movements of the diaphragms in their own plane while the fourth displacement (function) represents movement normal to the diaphragm (such as warping). Effects known as axial constraint, load diffusion and shear lag are treated in a unified manner in this theory.

The most mathematical part of the paper consists in a study of the conditions for the existence and the form of the self-equilibrating orthogonal end load systems (eigenloads) which vary similarly along all generators. This analysis is an extension of results previously obtained by F. B. Hildebrand [Tech. Notes Nat. Adv. Comm. Aeronaut., no. **894** (1943); these Rev. **7**, 231] for the load diffusion and shear lag in flat panels and rectangular box beams. For an independent treatment along similar lines of the torsion problem of the cylindrical shell see also T. von Kármán and W. Z. Chien [J. Aeronaut. Sci. **13**, 503–510 (1946); these Rev. **8**, 119]. *E. Reissner* (Cambridge, Mass.).

Gillies, A. W. On the bending of an initially straight beam under arbitrary loading. Philos. Mag. (7) **37**, 813–830 (1946).

The general solution for the bending of an initially straight beam under arbitrary loading is obtained in a form which it is suggested is convenient in application and easy to remember. A number of examples are worked out to illustrate the application of the general solution to particular problems. The method may be regarded as an extension of Macaulay's method. *Author's summary.*

Freeman, J. G. Mathematical theory of deflection of beam. Philos. Mag. (7) **37**, 855–862 (1946).

The paper is concerned with the large deflections of a beam under a central lateral load. The support forces are taken to be always normal to the center line of the beam. Thus the problem becomes a modification of the classical elastica problem. The solutions are obtained in terms of nonelementary functions and resemble qualitatively the solutions associated with the elastica. *G. F. Carrier.*

Rozovskiĭ, M. I. On the problem of analytic representation of the deformation processes of structures composed of visco-elastic elements. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1947**, 301–305 (1947). (Russian)

Integro-differential stress-strain relations for visco-elastic materials exhibiting after-effect are established and applied to the lateral vibrations of a beam which is made of such a material and rests on a visco-elastic foundation likewise exhibiting after-effect. *W. Prager* (Providence, R. I.).

Beskin, L., and Rosenberg, R. M. Higher modes of vibration by a method of sweeping. J. Aeronaut. Sci. **13**, 597–604 (1946).

The authors consider the lateral vibrations of straight beams with nonuniform cross sections. When such beams are vibrating in a normal mode, the deflection is $y(x)\cos(\omega t+\epsilon)$, where y satisfies the equation

$$\frac{d^2}{dx^2}\left(EIg\frac{d^2y}{dx^2}\right)=\omega^2wy;$$

E is Young's modulus, I is the moment of inertia of the cross section, g is the acceleration due to gravity and w is the mass per unit length. This equation has characteristic values $\omega_1, \omega_2, \cdots$ and characteristic functions $Y_1, Y_2, \cdots$. The function Y_n describes the nth mode of vibration; ω_n is the frequency of this mode.

To determine Y_1 the Stodola iteration process is used. In this process a guess y_1 is made for Y_1. This guess is inserted in the right side of the equation and an improved expression for Y_1 is obtained by numerical integration. This process is repeated until Y_1 is obtained with sufficient accuracy. A guess y_2 is then made for Y_2, and the first mode is "swept" from this guess by subtracting from it the leading term in the expansion of y_2 in a series in the orthogonal functions Y_n. The Stodola process is applied to the result. By alternating between this sweeping and the Stodola process, convergence to Y_2 is obtained. If the sweeping were omitted from this procedure, convergence to Y_1 would result. The functions $Y_3, Y_4, \cdots$ are found similarly.

The frequency ω_1 is found by equating the maximum strain energy in the first mode to the maximum kinetic energy in this mode. Similarly for $\omega_2, \omega_3, \cdots$. The method is also applicable to the torsion of beams of nonuniform cross section. *G. E. Hay* (Ann Arbor, Mich.).

Smolitzky, Kh. L. Le problème limite de la théorie de l'élasticité pour un cylindre infini. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 391–394 (1947).

The author discusses a solution of the wave equation in cylindrical coordinates in terms of an integral representation which contains exponential functions and Chebyshev polynomials under the integral. An application of this type of solution to the equations of elasticity is indicated.

A. E. Heins (Pittsburgh, Pa.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Ma, S. T. Equivalence of the Riesz method and the λ-limiting process for the classical electromagnetic field of a point source. Physical Rev. (2) **71**, 787–792 (1947).

As the title of this paper indicates, it is shown that Riesz's method of analytical continuation [see, for example, N. Fremberg, Proc. Roy. Soc. London. Ser. A. **188**, 18–31 (1946); these Rev. **8**, 302] and the λ-limiting process of Wentzel, Dirac and Pauli [see W. Pauli, Rev. Modern Physics **15**, 175–207 (1943); these Rev. **5**, 277] lead to the same results for the classical electromagnetic potential and its derivatives of a point source. *C. Kikuchi.*

Baudot, Jacques. Sur la forme matricielle des équations de Maxwell. C. R. Acad. Sci. Paris **224**, 1622–1624 (1947).

Mertens, Robert. On the diffraction of light by standing supersonic waves. Simon Stevin **25**, 62–68 (1947).

A narrow monochromatic light-beam goes through a liquid-filled vessel of rectangular cross section. If the liquid is disturbed by high frequency sound waves (supersonics) the emerging light-beam is split into a series of beams traveling in different directions with different intensities [see L. Brillouin, Ann. Physique (9) **17**, 88–122 (1921), in particular, p. 103]. An elementary theory has been proposed by C. V. Raman and N. S. Nagendra Nath [Proc. Indian Acad. Sci. Sect. A. **2**, 406–412, 413–420 (1935); **3**, 75–84, 119–125, 459–465 (1936)] for large values of ultrasonic wave lengths in the case of progressive sound waves and normal incidence. The author investigates the case of diffraction by standing sound waves. *A. E. Heins.*

Heins, Albert E., and Carlson, J. F. The reflection of an electromagnetic plane wave by an infinite set of plates. II. Quart. Appl. Math. **5**, 82–88 (1947).

In part I of this paper [Carlson and Heins, same Quart. **4**, 313–329 (1947); these Rev. **8**, 422] the authors calculated rigorously the reflection and transmission coefficients due to the incidence of plane monochromatic electromagnetic waves on an infinite set of staggered plates, the incident field being polarised parallel to the edges of the plates. They now consider the case when the incident field is polarised perpendicular to the edges of the plates. The method used is the same as that of part I. The problem is reduced to that of solving a rather complicated Wiener-Hopf integral equation. This integral equation is solved by Fourier transform methods and the reflection and transmission coefficients are found by considering the reflected and transmitted waves at a large distance from the edges of the plates. *E. T. Copson* (Dundee).

Wilkes, M. V. The oblique reflexion of very long wireless waves from the ionosphere. Proc. Roy. Soc. London. Ser. A. **189**, 130–147 (1947).

A general discussion of the field equations of magneto-ionic theory, assuming very long waves incident obliquely on the ionosphere in presence of a vertical magnetic field, leads to the conclusion that the ionosphere may be divided into a lower transition region and an upper reflecting region. In the reflecting region the field equations are

$$\partial^2 L/\partial\zeta^2+(\alpha+\zeta)L+\beta M=0, \quad \partial^2 M/\partial\zeta^2+(\alpha-\zeta)M+\beta L=0,$$

where L, $M=E_x\pm iE_y$, α and β are constants which depend only on the angle of incidence and it has been assumed that the ionization increases linearly with height. By introducing certain Laplace transforms the solution of these equations is shown to depend on solving an auxiliary equation $u''+2(\alpha+t^2)u'+\beta^2 u=0$; this equation is studied in detail. *M. C. Gray* (New York, N. Y.).

Banerjea, B. K. On the propagation of electromagnetic waves through the atmosphere. Proc. Roy. Soc. London. Ser. A. **190**, 67–81 (1947).

The author gives a general formulation of the equations of motion of an electron in the earth's magnetic field H under the influence of an electromagnetic field (incident electric vector $Ee^{i\nu t}$), without any restrictive assumptions on the relative directions of E and H. Similar treatment of Maxwell's equations in a homogeneous ionospheric region is followed by a discussion of the various theories developed by other writers and the different forms of solution are shown to be merely a consequence of the different frames of reference used. For convenience in comparison the author gives a symbol-correspondence chart for the symbols used by different authors. Polarization in the ionosphere is discussed in detail and the apparently unrelated conditions that have been postulated for reflection of electromagnetic waves at an ionospheric layer are shown to be fundamentally equivalent. *M. C. Gray* (New York, N. Y.).

Levin, M. L. A contribution to the theory of antennae. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 595–597 (1946).

The antenna is a thin perfect conductor of length l and variable radius $a=a_0 f(s)$. If an external electromotive force $K(s)$ is applied to the conductor parallel to the axis the complex Poynting theorem may be written

$$\int K(s)J^*(s)ds=\int_{S_R}\frac{c}{4\pi}[E,H^*]dS+2i\omega\int_{V_R}\frac{HH^*-EE^*}{8\pi}dV,$$

where V_R is the volume of the region bounded by the sphere S_R and the antenna. When $R\to\infty$ the surface integral is real and hence the reactive power is fully determined by the volume integral. If Q_0 is the reactive component for a cylindrical conductor of radius a_0 the corresponding Q for a conductor of variable radius (inscribed in the cylindrical one) may be found from

$$Q-Q_0=-(2k/c)\int_l\{|\psi(s)|^2-k^{-2}|d\psi/ds|^2\}(\log f)ds,$$

where $\psi(s)$ is the distribution function for the current along the axis of either conductor. *M. C. Gray.*

Bouwkamp, C. J., and de Bruijn, N. G. The electrostatic field of a point charge inside a cylinder, in connection with wave guide theory. J. Appl. Phys. **18**, 562–577 (1947).

The author studies the electrostatic field due to a point charge inside an infinitely long hollow circular cylinder. First it is shown that the surface charge density may be calculated. From this one can calculate the potential and the field. A second possibility is to solve the resulting boundary value problem directly and from this solution to find the interesting physical quantities. Finally, the method of Green's function is discussed; this appears to be the most direct manner by which one may calculate the solution.

There is a brief discussion of the acoustic radiation from a point source inside an infinite hollow cylinder. Attention is given to the problem of adapting the analytical results for numerical computation. *A. E. Heins* (Pittsburgh, Pa.).

Jouguet, Marc. Sur la propagation des ondes électromagnétiques dans les tuyaux courbés. C. R. Acad. Sci. Paris 222, 440–442 (1946).

Jouguet, Marc. Sur la propagation des ondes dans les tuyaux courbés. C. R. Acad. Sci. Paris 222, 537–538 (1946).

Jouguet, Marc. Sur l'influence de la courbure d'un guide d'ondes sur la propagation. C. R. Acad. Sci. Paris 223, 380–381 (1946).

Raymond, F.-H. Contribution à l'étude de la propagation sur une ligne hétérogène. J. Phys. Radium (8) 7, 171–177 (1946).

This paper continues the author's previous researches on the propagation of electromagnetic signals along various circuits (in particular, coaxial cables). Two rigorous solutions, of different forms, of a steady pulsation $p=j\omega$ in an arbitrary circuit are obtained. The author is led to a general definition of the coefficients of reflection and of transmission in a heterogeneous circuit. Utilizing Laplace transforms, he shows his solutions capable of extension to the study of transitory systems. No applications to technical problems are given. *A. L. Foster* (Berkeley, Calif.).

Kronsbein, John. The effect of insulating and conducting shields and partly stopped-off electrodes on current distribution in electrolytic cells. Proc. London Math. Soc. (2) 49, 260–281 (1947).

Several particular cases of controlled electrolytic plating of the inner surface of a metallic circular cylinder are studied here by means of conformal mapping. The cases considered are those in which the anode is a parallel circular cylinder placed in various positions inside the first cylinder, with insulating or conducting shields in the form of segments of cylindrical surfaces, or flat strips in the plane through the axes of the cylinders. The distribution of the steady current over the inner surface of the outer cylinder, which controls the rate of deposit of metal there, is determined by transforming the problem into one of the distribution of steady current between two coaxial cylinders. The transformations involve elliptic functions. Graphs of the current distributions for the various cases are shown. *R. V. Churchill.*

Quantum Mechanics

Wintner, Aurel. Stability and spectrum in the wave mechanics of lattices. Physical Rev. (2) 72, 81–82 (1947).

The author states that for an equation such as (Schrödinger's) $\psi''+(E-U)\psi=0$, the following two sets of numbers are the same, in case U is a periodic function: (1) the allowable values of E from a certain Hilbert space viewpoint, (2) the values of E for which $H(E)$ is real, where $H(E)$ is given by the statement that the general solution of the equation is essentially $\psi(x)=c_1e^{iH(E)x}\pi_1(x)+c_2e^{-iH(E)x}\pi_2(x)$, π_1 and π_2 being periodic and independent of c_1 and c_2.

I. E. Segal (Princeton, N. J.).

Betchov, Robert. La conservation de l'énergie et de la quantité de mouvement en mécanique statistique. Arch. Sci. Phys. Nat., Geneva (5) 29, 5–41 (1947).

Hylleraas, Egil A. Evaluation of transition probabilities for non-Coulombian central fields. Arch. Math. Naturvid. 48, no. 4, 57–88 (1945).

The basic idea in this paper is to use for the wave function describing the state of an atom one which has the correct asymptotic behavior at infinity required by the observed term value. Thus, for the S-state of an electron in a non-Coulombian field, the function represented by the following contour integral has the required property:

$$(*)\qquad y_n=\frac{n}{r}\frac{1}{2\pi i}\int_1^{(0,+)}\exp\left(-\frac{r}{2n}\frac{1+s}{1-s}\right)\frac{ds}{(-s)^{n+1}},$$

where the path of integration starts from $s=1$ on the upper side of the real s-axis and returns to the same point from the lower side after encircling the origin, once, in the positive direction. In equation (*), n is in general not an integer, but depends on the (experimentally) known energy of the state. For nonintegral n the function represented by the contour integral (*) has a singularity at the origin, behaving like $-(\pi/r)\sin n\pi$. For the purposes of evaluating the matrix elements giving the transition probabilities, this singularity is not regarded as an unsatisfactory feature.

For the P-states the representation

$$(**)\qquad z_m=\frac{r}{m^3}\frac{1}{2\pi i}\int_1^{(0,+)}\exp\left(-\frac{r}{2m}\frac{1+t}{1-t}\right)\frac{dt}{t^{m-1}(1-t)^4},\qquad m=2,3,\cdots,$$

is used. Only integral values for m are considered since the Rydberg corrections for the P-states are generally very small compared to those for the S-states.

With the wave functions given by (*) and (**) the integrals $N_n=\int_0^\infty y_n^2r^2dr$ and

$$(n,m)=\frac{4m^2}{1-(m^2/n^2)}\int_0^\infty z_m\frac{dy_n}{dr}r^2dr,$$

with n nonintegral and $m=2,3,\cdots$, are evaluated. Thus

$$N_n=2n^3\left\{1-\left(\frac{\sin n\pi}{\pi}\right)^2\left[\sum_{k=0}^{\infty}\frac{1}{(n+1+k)^2}-\frac{1}{n}+\frac{1}{2n^2}\right]\right\}.$$

Certain special properties of confluent hypergeometric functions required in the evaluation of the integrals N_n and (n,m) are treated separately in a series of "mathematical notes." The behavior of the solutions of the differential equation

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{1}{4n^2}+\frac{1}{r}\right)\psi=0$$

at $r=0$ and their expansions for nonintegral n are considered in particular detail. *S. Chandrasekhar.*

Schönberg, Mario. Quantum theory of the point electron. I. Anais Acad. Brasil. Ci. 18, 297–339 (1946).

An attempt to quantize the author's classical electron theory [Physical Rev. (2) 69, 211–224 (1945); Summa Brasil. Math. 1, 41–75, 77–114 (1946); these Rev. 8, 427, 428]. Due to the introduction of advanced and retarded potentials from the outset, special attention must be paid to the role of the time in quantum dynamics. A q-number time is used with eigenvalues t. The connection between the

other observables at two different t's is described by an operator which is no longer unitary. The problem of completely specifying this operator is not solved. There are still Schroedinger equations for the states, but they lose their customary interpretation. A derivation of the invariant electromagnetic matrix elements along these lines is given. *A. Pais* (Princeton, N. J.).

Lopes, J. Leite. On the divergences of quantum electrodynamics. Anais Acad. Brasil. Ci. 19, 51–66 (1947).

Ideas suggested in earlier notes by M. Schönberg and the author [Physical Rev. (2) 67, 122–123 (1945)] are developed in calculating the self-energy of a point charge. The Lagrangian of the field is taken to be

$$L_0=-\tfrac{1}{8}\sum_{\mu\nu}\left\{\frac{\partial\varphi^{\nu}_{\mathrm{ret}}}{\partial x_\mu}-\frac{\partial\varphi^{\mu}_{\mathrm{ret}}}{\partial x_\nu}\right\}^2+\tfrac{1}{8}\sum_{\mu\nu}\left\{\frac{\partial\varphi^{\nu}_{\mathrm{adv}}}{\partial x_\mu}-\frac{\partial\varphi^{\mu}_{\mathrm{adv}}}{\partial x_\nu}\right\}^2$$

with the interaction term

$$L_{\mathrm{int}}=\tfrac{1}{2}\sum_{\nu}(\varphi^{\nu}_{\mathrm{ret}}-\varphi^{\nu}_{\mathrm{adv}})S_\nu.$$

The field is quantized in the usual way. It is shown that the contribution of the advanced field to the electrostatic self-energy just cancels that due to the retarded field. The transverse self-energy terms, however, do not cancel each other. *C. Kikuchi* (East Lansing, Mich.).

Eliezer, C. Jayaratnam. Quantum electrodynamics and the interaction of hydrogen-like atoms with a radiation field. Bull. Calcutta Math. Soc. 38, 145–150 (1946).

Using Dirac's new theory of electrodynamics the author studies the problem of an electron in a Coulomb field interacting with the radiation field. It is shown that the wave function is free from divergences to any order of approximation in the perturbation theory in which the radiation interaction is taken as a perturbation, provided the λ-limiting process and the negative energy photons introduced by Dirac are employed and the particular solution for the wave function corresponding to outgoing waves is taken. If the wave function is written in a series $\psi=\psi_0+\psi_1+\psi_2+\cdots$ according to the power of the radiation operator involved and if $\psi_{n,m}$ denotes that part of ψ_n which contains m photons, then $\psi_{n,m}$ is shown to be zero for all values of m different from n. The author concludes from this that radiation damping does not cause any displacement of spectral lines nor does it influence other radiative processes such as the photo-electric effect and Bremsstrahlung. *S. Kusaka.*

Tonnelat, Marie-Antoinette. Théorie euclidienne de l'électromagnétisme et de la gravitation. Disquisit. Math. Phys. 3, 249–274 (1943).

The author extends de Broglie's neutrino theory of light to the description of both electromagnetism and gravitation. The product representation of four spinors is shown to be equivalent to one representation of spin 2, three of spin 1 and two of spin 0. The first is interpreted as describing gravitational quanta, the second photons and the last is attributed to forming some modifications of the gravitational theory. Since all the components of the tensors satisfy the usual second order linear differential equation, the identification with the theory of gravitation is made by considering only the quasi-Euclidean approximation in which the gravitational effects are assumed to be weak. It is shown that a space such as that of De Sitter where the cosmological constant is not zero corresponds in the present theory to gravitational quanta having finite rest mass of the order of 10^{-64} gm. *S. Kusaka.*

Novacu, Valer. On the undor form of the equations of the photon. Bull. Sci. École Polytech. Timişoara 10, 133–145 (1941). (Romanian)

Novacu, Valer. On the interaction between photons and electrons. Bull. Sci. École Polytech. Timişoara 10, 331–344 (1941). (Romanian)

Grosjean, P.-V. Solutions autocomplémentaires et bipolarisées d'équations spinorielles. Bull. Soc. Roy. Sci. Liége 15, 220–235 (1946).

The paper begins with a summary of some known facts about the Dirac equations which are written in two-component spinor form as

$$(*)\qquad \left(S^0\frac{\partial}{\partial t}+S^i\frac{\partial}{\partial x^i}\right)\psi=ik_0\varphi,\qquad \left(S^0\frac{\partial}{\partial t}-S^i\frac{\partial}{\partial x^i}\right)\varphi=ik_0\psi,$$

where S^0 is the unit matrix in two dimensions and S^i are the Pauli spin matrices. The author then defines β^4 as the unit four-dimensional matrix and takes β^i as a particular four-dimensional representation of the Pauli spin matrices. He then defines "the quasi-Maxwell equations" as those obtained from (*) by replacing S^μ by β^μ ($\mu=1, 2, 3, 4$). He shows that the new system reduces to the system (*) for suitably restricted ψ and φ. It is shown that the new system is not related to the Maxwell equations but is a subset of the de Broglie equations for the photon. *A. H. Taub.*

Potier, Robert. Sur les systèmes d'équations d'onde mettant en jeu une représentation irréductible du groupe de Lorentz. C. R. Acad. Sci. Paris 224, 1332–1334 (1947).

The author has previously proposed [same C. R. 222, 638–640 (1946); these Rev. 7, 447] a system of equations involving an irreducible representation of the Lorentz group to represent a particle of arbitrary spin. In this paper he shows that these equations contain no auxiliary conditions and determines the z-component of the spin operator's proper values. He also discusses monochromatic plane waves in the coordinate system in which the particle is at rest.

A. H. Taub (Princeton, N. J.).

van Isacker, Jacques. Généralisation de l'équation d'une particule de spin maximum 1. C. R. Acad. Sci. Paris 224, 1758–1760 (1947).

Using de Broglie's theory of particles with higher spin, the wave equation for particles with maximum spin of unity is written down. This is then expressed in Kemmer's form with the generalization that the constant mass term is replaced by a matrix with eigenvalues corresponding to different values of the mass for the spin one and spin zero particles. It is then shown that the matrix operator corresponding to the difference of these mass values is, in the nonrelativistic limit, proportional to the square of the spin operator. This result is interpreted to mean that the mass difference arises from the inertia due to the spin.

S. Kusaka (Princeton, N. J.).

DECEMBER ISSUE IS AN INDEX WHICH HAS BEEN PHOTOGRAPHED AT THE BEGINNING OF THE VOLUME(S).